An Integrated Model of University Endowments*

Georg Cejnek  
WU Vienna University of Economics and Business  
Richard Franz  
WU Vienna University of Economics and Business  
Neal M. Stoughton  
WU Vienna University of Economics and Business and Vienna Graduate School of Finance

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Abstract

University endowments have attracted considerable recent attention as an important institutional investor class while developing a unique aspect to their investing policies – often referred to as the “endowment model”. Our paper develops a formal theoretical endowment model within which to view the consistent application of asset allocation and spending policies. The innovation in this research is that we tie the donations process to the investment performance. An empirical analysis using recent endowment data validates our donations process assumptions. We investigate both substitution and wealth effects and find that conventional predictions can be reversed due to this form of donations endogeneity. Specifically risky asset allocations are higher for endowments with more donations despite the substitution effect and spending rates for smaller endowments are more volatile than for larger endowments due to the wealth effect of endowment size. Looking at empirical data on actual US endowment practices, we find that universities behave as though the substitution effect persists, although the size impact on endowment practices is more in line with our model.

1 Introduction

The popular “endowment model” of universities consists of the following essential ingredients: (1) a long investment horizon; (2) high degrees of diversification; (3) active management as exemplified by manager selection criteria; (4) a bias towards international equities; (5) minimal levels of fixed income and cash; (5) a heavy allocation to alternative investments such as hedge funds and private equity;

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a bias towards illiquid assets and (7) a relatively smooth expenditure policy. We develop in this paper a formal theoretical model within which to investigate optimal endowment policies such as asset allocation, spending and the impact of endowment size. The key innovation of our work is that we consider the impact of a donations process which provides a feedback effect on optimal university endowment management. In other words, the nonendowment income process is not exogenous in our model; instead it is influenced by the investment outcomes. We also test our predictions on actual US university endowment data investigating the relationships between risky asset allocations, investment opportunities, donation impacts and spending levels.

University endowments as well as endowments for many nonprofit foundations and churches have provided valuable support for centuries. Today, a number of institutions with premier reputations such as Harvard, Yale, Stanford, etc. have endowments in the tens of billions of dollars and have enjoyed significant attention and success as an important class of institutional investor\textsuperscript{1}. These endowments often provide over 20% of a university’s operating budget – sometimes more than tuition fees in the case of private schools. It is frequently stated that the purpose of an endowment is to smooth income flows and provide independence from politically driven funding constraints (e.g., support from state governments). But also endowments can have an important role in university affairs by creating an ownership society in which alumni and other important university constituents maintain a stake in the reputation of a university. The endowment provides a conduit by which donations are encouraged and the base of contributors is broadened. For instance university ranking schemes such as that of \textit{US News} consider alumni giving as an explicit component of the rating scale employed. It is also common for key donors to be invited to join the governance structure of university endowments such as membership on the investment committee where they take part in decisions regarding investment management and the strategic allocation of funds.

Optimal endowment policy considering donations and stochastic tuition fees was first discussed by Merton \citeyear{Merton1993} but until recently very little has been done since to investigate these feedback effects within a fully optimal model. In the Merton framework, a university should consider both its endowment as well as nonendowment income. If the nonendowment income is represented by a full set of tradeable assets, then this source of income can be valued using the asset prices and added to obtain a fully inclusive value. Merton then defines both a \textit{substitution} effect as well as a \textit{wealth} effect. The former effect results in lower asset allocation to risky assets as a fraction of just the endowment wealth, as long as the nonendowment income is positively correlated with the return to risky assets. This is basically an application of the well-known “hedging demands”. The latter (wealth) effect, as defined by Merton results in a higher overall risk exposure. Combining the substitution and wealth effect can result in ambiguous predictions depending on which effect dominates. In terms of testability, however, if one focuses purely on the substitution effect the predictions of this model are therefore that universities with higher donation income have lower risky asset allocations as a percentage of their endowment value. As an example, if a key donor makes gifts that are correlated with stock returns in the pharmaceutical industry, such an endowment might optimally have a short position in pharmaceutical stocks

\footnote{\textsuperscript{1}See for instance the influential book of Swensen \citeyear{Swensen2009}.}
according to the Merton model.

In our paper, we investigate both a substitution effect and a wealth effect. However we do not consider only the case where nonendowment income (donations) are represented by complete markets. Further, we consider income that is nonlinearly related to investment returns, so that income and investment returns are no longer perfect substitutes. Finally, our major innovation is to consider a unique feedback effect between endowment performance and donation income. That is, donation income is assumed to be positively influenced by endowment returns so that the income streams are no longer separable from one another. Moreover we consider the impact not only on asset allocation but university spending as well. We find that there are circumstances where the predictions of the Merton model can be reversed, both in terms of substitution and wealth effects. Our results have important implications for how endowments are managed across the spectrum of universities.

We consider two basic forms of donation policy: (1) donations are positively related with the return to the risky assets that the university invests in, and (2) donations are positively related with the performance as measured by rate of return to the endowment fund itself. We believe these two forms of donation process can be supported by observations from recent university endowment experience. For instance many endowments hold assets in venture capital funds and private equity of firms run by their alumni – who are also key donors. Our second donations process is motivated by the way in which key donors often influence endowment management decisions by serving on boards of directors. Anecdotal evidence indicates that such donors ask the endowment managers whether they are earning returns equal to or greater than what such wealthy donors can earn on their own investments. If not, they may suggest that the university is better off letting them defer their donations to a later date, when either endowment management has improved or the private investment opportunities of donors have decreased. Using a six year sample of actual data on donations and endowment returns we find strong empirical support for both of these (nonlinear) donations specifications.

We embed these two forms of donation processes into an intertemporal consumption-portfolio problem with otherwise standard assumptions. Universities are modeled as infinitely lived agents, with recursive utility as pioneered by Epstein and Zin (1989). They derive all of their utility from a stream of spending inflows to the budget from the endowment appreciation. Investment opportunities are represented by a single risky asset and a nonstochastic riskfree asset. The risky asset has a risk premium which is time-varying and follows the structure of the AR(1) process assumed in Campbell and Viceira (1999). Our important results are obtained in comparison to the model of Campbell and Viceira (1999) where donations are fixed at zero. We identify, as did Merton (1993), the existence of a perfect substitution effect when donations are perfectly correlated with risky asset returns. This result extends to the case where overall endowment returns are perfectly correlated with donations. We find that donations have no effect on the spending rate out of endowment value, however. This is because although donations increase endowment wealth, spending increases proportionately, so that the stochastic growth rate of both spending and endowment wealth is the same as without donations.

We then go on to extend the analytic results of linear donation flows by considering a case where donations are related to the positive part of the rate of return process for both the risky asset as well as
the overall endowment return. This implies that the university has a nonlinear option-type donations process which generates some nonlinearity in the asset allocation as a function of investment opportunities. Both of these donation processes have important implications for the consumption to wealth ratio as a function of investment opportunities. When investment opportunities are poor (e.g., after an unusually high appreciation of the market returns), the rate of consumption is a higher percentage of wealth than in the case without donations. On the other hand, when investment opportunities are good the rate of spending to the value of the endowment increases, but less so than in the case without donations. In fact, it converges to the no-donations case. The result of both of these donations processes is that there is significantly less intertemporal variation of the spending to value ratio with donations as compared to the situation without donations. This can therefore account for some of the observed behavior whereby universities do not vary their spending in one-to-one relation with the endowment value – a result at variance with the original Merton [1971] myopic solution.

Our model is then extended to consider the impacts of endowment size. We do this by modifying the assumption that donation inflows depend on net endowment size, i.e., after spending is deducted, and instead depend on the beginning-of-period endowment value. As before donations are related to rates of return on the risky asset or the overall endowment when this is positive. This provides a motive for smaller endowments to grow the size of their endowments in order to achieve even higher donations in the future. We now find some important impacts from this wealth effect, in addition to the substitution effect mentioned earlier. First of all, the asset allocation relation to investment opportunities now becomes convex. This convexity effect is more pronounced for the case where donations are related to the overall endowment returns. It implies that now the asset allocation towards risky assets is higher than in the case without donations, especially for good but also as well for low investment opportunity situations. This asset allocation convexity effect for high investment opportunities is negatively related to endowment size, i.e., smaller endowments have a tendency toward going for higher risky asset allocations in order to gain future donations inflows than do larger endowments. We also find that the minimum consumption to wealth ratio is not constant as in the case without donations, but increases with the size of the endowment – implying a justification for higher spending patterns observed by universities with large endowments.

We also bring our predictions to the data. We find some support for the predictions of our optimal model; endowments do associate their asset allocations toward risk when investment opportunities are more favorable. There does appear to be a substitution effect based on our model of intensity of the donations feedback effect on average. There is a strong size effect in the data, exemplified as a higher propensity to take risks, while the spending rate out of endowment wealth is lower but more variable for larger endowments. When we perform a multivariate analysis, we find that asset allocation practices of endowments are determined mostly by investment opportunities and size. Spending rates are again strongly related to size and investment opportunities through asset allocations. The level of donations also comes through as intensity of donations ameliorates any reductions in spending to some extent.

Dybvig [1999] considers the impact of a monotonicity constraint on spending over time and shows that this changes university endowment policies considerably. However this constraint is imposed, rather than being derived from fundamental principles.
While our paper is not the first to consider an auxiliary source of income in a consumption/portfolio problem, it is the first to model an explicit relationship between investment performance and auxiliary income (donations). For instance in Benzoni, Collin-Dufresne, and Goldstein(2007) cointegrated labor income is considered in a finite horizon problem. The purpose there is to support existing observations whereby agents utilize an inverse U-shaped risky asset holding patterns throughout their lifetime. By contrast in our model the growth in the university endowment due to higher returns drives a higher magnitude of the donations process, and this effect needs to be accounted for in the asset allocation optimization.

Another paper which includes endogenous donations is that of Gilbert and Hrdlicka(2014). In their model, a university holds social assets and financial assets. The university can shift resources between these two forms of asset holdings. When donations are related to the university’s investment in social assets and the university trustees maximize social dividends produced by social assets, then endowments are small and only those universities with unproductive social returns hold large endowments. Moreover endowments are often achieving only riskless returns because of precautionary motives. By contrast in our model, we are able to accommodate large holdings of risky assets alongside risky donations even with more conventional preferences, because donations are tied to endowment returns. We also focus more on the optimal spending paths.

It is clear also that our positive dependent donations process corresponds to the literature on mutual fund inflows chasing performance and on the mutual fund tournament effect (Chevalier and Ellison, 1997) and (Sirri and Tufano, 1998). Nevertheless, what is different in our model as compared to mutual funds is the endogenous need for spending, and for the size dependence. Although there is limited empirical evidence on donations and performance, Oster (2001) reports that in 1998 80% of overall endowment donations were provided by 10% of all donors. In her model, Oster (2001) considers altruistic motives since donors derive utility from private consumption, the donation itself and public goods provided by the university. She suggests that very large donors derive more utility from donations themselves (sometimes referred to as a warm glow effect) than from the provisions of public goods.

Dimmock (2012) investigates background risk – defined as volatility of nonendowment income. He finds a negative dependence between higher volatility nonendowment income and allocations to risky assets. This is potentially consistent with Merton’s substitution hypothesis, although the necessary correlation structure is not tested. Blanchett (2014) does look at correlations between asset class returns and aggregate donation inflows for different classes of philanthropic organizations. He solves a risk-based portfolio optimization problem and simulates the results. However a fully dynamic model incorporating investment opportunities, spending and the feedback effect of donations is not addressed. Barber and Wang (2013) investigate an attribution analysis involving university endowments and find positive alphas relative to domestic stocks and bonds for larger endowments, elite institutions and those with high reputations as measured by SAT scores. However, when additional asset categories such as hedge funds and private equity are considered the positive alphas disappear. The evidence contained therein is broadly consistent with higher exposure to risky assets for larger endowments and
a complementary rather than substitution effect based on the diversity of asset allocations. Ang, Ayala, and Goetzmann (2013) use asset allocation weights as a method of deriving implied beliefs about the alphas of asset classes and conclude that alternative investments must have been perceived to have higher than normal returns.

In addition to the papers cited above other relevant literature is as follows. Campbell, Cocco, Gomes, Maenhout, and Viceira (2001) solve a similar model without donations numerically. Bhamra and Uppal (2006) derive a closed form solution for an intertemporal consumption portfolio problem in a three period model. Satchell and Thorp (2007) and Satchell, Thorp, and Williams (2012) analyze endowment spending with recursive utility and non-normal return distributions. They find the elasticity of intertemporal substitution to be the main factor determining optimal spending policies. However their papers assume donations are unrelated to the optimal university endowment management policies and therefore are unable to ascertain the kind of feedback effects in our paper. Brown and Tiu (2012) report that many endowments change their spending policy frequently and that changes in spending policy often lead to subsequent changes in asset allocation. For a comprehensive survey of the entire academic research on university endowments, see Cejnek, Franz, Randl, and Stoughton (2014).

The remainder of this paper is organized as follows: Section 2 presents the model specification and results for the case without an endowment size effect. We also we present empirical support for our donations model. Section 3 extends our results to the case where endowments have an additional need to grow over time. Section 4 investigate our predictions on a sample of recent annual disaggregated endowment data. Section 5 presents an illustration by backtesting our model on a typical university endowment process. Section 6 concludes the paper.

2 Model Specification

This section introduces our model for dynamic consumption and investment decisions taking into account donation inflows. We model a university as an infinitely lived agent which derives utility from contributions to the operating budget. Utilizing a standard utility maximizing framework over consumption flows in an infinite horizon gives the interpretation of consumption as synonymous with spending.

The university derives its endowment from both returns on its investment portfolio as well as donations. For simplicity assume that there are two assets in which the endowment can be invested. The risky asset has gross returns $R_{1,t+1}$ from time period $t$ to $t+1$. The riskfree asset has gross returns $R_f$ over the same period. The riskfree rate is assumed to be nonstochastic over time. The endowment accumulation process is therefore given by

$$W_{t+1} = (\alpha_t(R_{1,t+1} - R_f) + R_f)(W_t - C_t) + D_{t+1},$$

where $C_t$ is spending at time $t$, $W_t$ is the value of the endowment at time $t$, $\alpha_t$ represents the asset

\[ \frac{1}{3} \text{This can be easily generalized to multiple assets depending on the stochastic returns structure.} \]
allocation to the risky asset as a percentage of beginning of period endowment wealth and $D_{t+1}$ are donations to the endowment over the period from time $t$ to $t+1$. We consider a number of different specifications for the donations process, $D_t$.

For purposes of computation it is helpful to define the total endowment gross return process, consisting of both investment and donations:

$$R_{m,t+1} = \frac{W_{t+1}}{W_t - C_t}.$$  \hspace{1cm} (2)

Note that as long as donations are positive, this overstates the usual return to the investment portfolio. The investment returns are given by

$$R_{p,t+1} = \alpha_t (R_{1,t+1} - R_f) + R_f.$$  \hspace{1cm} (3)

The university maximizes its intertemporal utility by choosing optimal consumption and allocation to the risky asset as given by the following recursive utility function:

$$U(C_t, E_{t+1}) = \max_{C_t} \left( (1 - \delta)C_t^{(1-\rho)/\theta} + \delta \left( E_{t+1}^{1-\rho} \right)^{\theta/(1-\rho)} \right),$$  \hspace{1cm} (4)

where $\delta$ is the time discount factor, the coefficient of relative risk aversion is $\rho$, $\psi$ is the coefficient of intertemporal elasticity of substitution, and $\theta = (1-\rho)/(1-\psi^{-1})$. The expression (4) derives from Kreps-Porteus preferences and is an analytically tractable form of preference aggregation. These preferences were first employed by Epstein and Zin (1989) and Weil (1990), and have been used by many authors since. It is more general than the usual Von Neumann Morgenstern time additive utility because it separates the risk tolerance from the intertemporal elasticity of substitution. In the asset pricing literature this specification along with market clearing of a representative investor gives an asset pricing model in which both consumption growth rates as well as market returns are used in deriving risk premia.

Since one purpose of our model is to capture stochastic investment opportunities we also employ the well-known asset return structure of Campbell and Viceira (1999). This involves the assumption that conditional risk premia of natural logarithm of risky asset returns, $r_{1,t}$ follow a mean-reverting AR(1) process:

$$E_{t+1} - r_f = x_t,$$  \hspace{1cm} (5)

where $x_t$ is assumed to be an observable process following the structure below:

$$x_{t+1} = \mu + \phi(x_t - \mu) + \eta_{t+1} = (1 - \phi)\mu + \phi x_t + \eta_{t+1}.$$  \hspace{1cm} (6)

The residual of this process, $\eta_{t+1} \sim N(0, \sigma^2_{\eta})$, is normally distributed. The log returns process can therefore be written as

$$r_{1,t+1} = x_t + r_f + \eta_{t+1},$$  \hspace{1cm} (7)

\footnote{Lower case letters refer to logarithms of their upper case counterparts as is standard notation.}
where the unexpected return $u_t$ is also normally distributed, $u_{t+1} \sim N(0, \sigma^2_u)$, and is correlated with the residual term in the AR(1) process of expected returns; $\text{cov}(u_{t+1}, \eta_{t+1}) = \sigma_{u\eta}$.

### 2.1 Donations Process

To compare our model with the existing literature, we begin with a simplified linear reduced form model of donations. We do this in order to show what features of the standard literature we can employ and in what directions our model can then be extended to obtain new results.

In that respect, suppose first that the donations process follows

$$D_{t+1} = d(R_{t+1} - R_f)(W_t - C_t), \quad (8)$$

for some (strictly positive) coefficient $d$. This process embodies two assumptions. First, donations flows are perfectly correlated with returns to risky assets. Second, donations are positively impacted by (beginning of period) endowment magnitude net of spending. We believe this form of positive correlation between donation inflows and contemporaneous capital market returns to be realistic, which stands in contrast to the setup of Satchell and Thorp (2007) and Satchell, Thorp, and Williams (2012), who assume zero correlation between donations and returns. Somewhat more controversially, this process assumes that greater spending, \textit{cet par} reduces donation flows. That is, the positive feedback effect on donations through investment performance is only through the financial aspect and not through spending activities. The specification of equation (8) has the additional unattractive feature that donations are negative when the risky asset underperforms the riskfree asset. Despite these deficiencies – which we address in further generalizations later – this formulation does allow for an easy analytic derivation of the substitution effect.

Second, let us consider a donations process that is perfectly correlated with the optimal endowment portfolio return. We assume that

$$D_{t+1} = d(R_{p,t+1} - R_f)(W_t - C_t). \quad (9)$$

This donation process depends on the realized returns of the optimal investment portfolio. So long as the optimal portfolio outperforms the riskfree asset, one sees positive donation flows. This form of donations process can be justified by the fact that major donors to universities are often present on endowment boards - indeed on the investment committee. This gives them both influence over how the endowment is invested, but also facilitates their interest in donating additional funds. Equation (9) can be regarded as an equilibrium condition in that if endowment returns underperform, donors have incentives to defer donations, if endowment returns outperform, donors are more willing to accelerate donations.

For both donations processes, equations (8) and (9), it is helpful at this point to describe the necessary conditions for the overall problem. First, we show in Appendix A that when the donations process is measurable in the rate of returns process, the value function, $V_t(x_t, C_t, W_t) = U(C_t, \text{EU}_{t+1} | x_t, W_t)$ is homogeneous of degree 1 in $W_t$. It then follows Epstein and Zin (1991) that the value function can be
expressed as

\[ V_t(x_t, C_t, W_t)/W_t = (1 - \delta)^{-\psi/(1-\psi)} \left( \frac{C_t}{W_t} \right)^{1/(1-\psi)}. \]  \hspace{1cm} (10)

Substituting this back into (4) then gives the following two necessary conditions (stochastic Euler equations):

\[ E_t \left( \delta \left( \frac{C_{t+1}}{C_t} \right)^{-1/\psi} R_{m,t+1} \right)^{\theta} = 1, \]  \hspace{1cm} (11)

and

\[ E_t \left( \left( \frac{C_{t+1}}{C_t} \right)^{-\theta/\psi} R_{m,t+1}^{\theta} \left( \frac{R_{1,t+1} - R_f}{R_{m,t+1}} \right) \right) = 0. \]  \hspace{1cm} (12)

The most important aspect of these two equations to note is that the effective stochastic discount factor involves the growth rate in consumption multiplied by the gross return on total wealth accumulation - not just the total investment return.

Using the optimal characterization of the consumption portfolio problem, we can now precisely specify the impact of donations in the form of the following proposition.

**Proposition 1.** For both donations processes, equations (8) and (9) there exists a pure substitution effect in which the allocation to the risky asset is reduced for all values of the state of investment opportunities by the same amount. Specifically, if \( \alpha_0 \) denotes the asset allocation in the absence of donations, the asset allocation in the presence of donations perfectly correlated with the return to the risky asset, equation (8) is given by

\[ \alpha = \alpha_0 - d, \]  \hspace{1cm} (13)

and the allocation when donations depend on the optimized portfolio return, equation (9) is given by

\[ \alpha = \alpha_0/(1 + d). \]  \hspace{1cm} (14)

Further, in both cases, the ratio of consumption to total wealth with donations is exactly equal to the ratio of consumption to wealth without donations.

**Proof.** Substituting equation (8) into equation (4) gives the following endowment accumulation process:

\[ W_{t+1} = ((\alpha_t + d)(R_{1,t+1} - R_f) + R_f)(W_t - C_t). \]  \hspace{1cm} (15)

The standard consumption portfolio problem consists of problem (4) subject to equation (15) with \( d = 0 \). Hence our first donations process leads to a problem which is isomorphic to the standard problem. If \( \alpha_0 \) is the solution to the problem with zero donations, it then follows from the optimality condition (12) that \( \alpha_t = \alpha_0 - d \) represents the solution to this donations problem.

Similarly, substituting equation (9) into equation (4) now gives the following endowment accumulation process for the second donations process:

\[ W_{t+1} = ((1 + d)\alpha_t(R_{1,t+1} - R_f) + R_f)(W_t - C_t). \]  \hspace{1cm} (16)
Analogously, \( a_0 \) represents the solution in the absence of donations \((d = 0)\), then setting \((1 + d)\alpha_t = a_0\), so that \(\alpha_t = a_0/(1 + d)\) gives a wealth accumulation process such that \(W_{t+1} = a_0(R_{1,t+1} - R_f)(W_t - C_t)\). Therefore for both donations processes, the wealth accumulation process as a function of \(a_0\), \(W_t\), and \(C_t\) is identical with the case without donations. In fact we can write the gross rate of return as

\[
R_{m,t+1} = \frac{W_{t+1}}{W_t - C_t} = a_0(R_{1,t+1} - R_f) + R_f.
\]

Substituting this into the Euler condition equation \((11)\), therefore implies that the distribution of \(C_{t+1}/C_t\) is identical with the case without donations. Hence the ratio of consumption to wealth is also identical.

Proposition \(1\) shows that a substitution effect is manifest in asset allocation, but not in the consumption/wealth ratio. Basically the exposure to the risky asset as a percentage of endowment value is reduced percent for percent by the donations inflows in the case of equation \((8)\) and is contracted by an identical factor in the case of equation \((9)\). These results are due to the strong assumption of perfect correlation, which we generalize below. In the case of endowment spending, it is worth pointing out that with donations, wealth increases at a faster rate. But spending also increases at the same faster rate, so that the ratio of one to the other is identical to that without donations.

### 2.2 Positive Donation Flows

It is apparent that while the assumed donation processes of \((8)\) and \((9)\) make it easy to compare the situation with and without donation flows, and thereby to derive the precise substitution effect, it has the disadvantage that one does not expect donations to be perfectly correlated with risky asset returns or endowment returns and further donations are never negative in reality. Hence we now consider the positive part of the previous donation flows. In these cases, we will have to resort to numerical solutions since the substitution effect cannot be derived analytically and varies with investment opportunities.

Analogous to equation \((8)\), suppose that the donations process follows

\[
D_{t+1} = \max\{d(R_{1,t+1} - 1)(W_t - C_t) + \epsilon_t(W_t - C_t), 0\},
\]

for some random variation, \(\epsilon_t \sim N(0, \sigma^2)\). It is apparent that in this case even if \(R_{1,t+1}\) is lognormally distributed, the positive part of the excess return, as in any call option, is not lognormally distributed. Therefore the approximate solution method of Campbell and Viceira (1999) does not work.

Interestingly, however, the value function is still homogeneous of degree 1 in endowment value, \(W_t\). This means that the value function still satisfies \((10)\) and the stochastic Euler conditions are as before embodied in equations \((11)\) and \((12)\). See Appendix for the verification of this property.

Our second positive donations process assumes, analogous to \((17)\) that the donations are positively related to the excess returns on the investment portfolio, as long as they are positive, namely

\[
D_{t+1} = \max\{d(R_{p,t+1} - 1)(W_t - C_t) + \epsilon_t(W_t - C_t), 0\},
\]
Both of the processes in equations (17) and (18) imply that the policy variables will be independent of wealth. Hence there is no wealth effect. To capture more generality we also consider two more donations processes.

\[ D_{t+1} = \max \{ d(R_{1,t+1} - 1)W_t + \epsilon_t W_t, 0 \}. \]  (19)

Now donations are modeled as directly related to rates of returns on the risky asset without considering the prior periods spending policy. This gives an indirect greater weight to size, as the donations inflows are not reduced by higher spending policies. Thus, our perspective now has shifted to indicate that spending not only benefits the university in terms of consumption, but it also enhances reputation which improves donation inflows. We again include the additional noise term, \( \epsilon \), to model an imperfect correlation between donations and the risky rate of return.

Analogous to equation (18) we also consider the related donations process which is a positive function of the rate of return to the optimized investment portfolio instead of just the risky asset. This process is:

\[ D_{t+1} = \max \{ d(R_{p,t+1} - 1)W_t + \epsilon_t W_t, 0 \}. \]  (20)

Once again donations are imperfectly correlated because of the exogenous noise.

### 2.3 Empirical Support

In order to support our assumed donations processes, we employ a representative sample of US university endowments that report annual returns, endowment size, spending levels and donation inflows to NACUBO from 2008 to 2013.

We have direct evidence on each university’s endowment returns, \( R_{p,t+1} \). In order to obtain a proxy for each endowment risky asset returns process, \( R_{1,t+1} \), we use data on actual endowment asset allocations. NACUBO reports actual allocations to 12 asset classes for all individual endowments each year: domestic equities, fixed income, international equities, real estate, venture capital, private equity, managed futures, marketable alternative strategies, distressed debt, other alternative strategies and short-term assets/cash. We approximate appropriate risky asset returns using only two benchmark returns, which are total returns on US stocks (S&P 500 Index) and US long-term bonds (Barclays US Treasury 7-10 Year Index). We multiply those benchmark returns with the corresponding asset allocation weights, aggregating together domestic equities, international equities, alternatives, distressed debt, real estate, venture capital, private equity, managed futures, in the risky class and everything else (fixed income and short term/cash) as a riskless class. This ensures that our risky asset return proxies exhibit some heterogeneity across endowments, reflecting varying policy portfolios in the cross section of endowments, while they are distinct from the actual endowment returns.

We test eight donations specification processes. The first four are based on a linear relationship.

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\( ^5 \)NACUBO/Commonfund is an organization of College and University Business Officers that has been collecting survey data for many years and publishes an annual report.
First, we use two linear specifications corresponding to equations (17) and (18):

\[
\frac{D_{t+1}}{W_t - C_t} = d(R_{1,t+1} - 1) + \epsilon \\
\frac{D_{t+1}}{W_t - C_t} = d(R_{p,t+1} - 1) + \epsilon.
\]

Here donations are divided by net endowment size (after spending). This specification is homogeneous with respect to wealth. Second, we employ two specifications with a wealth effect from equations (19) and (20):

\[
\frac{D_{t+1}}{W_t} = d(R_{1,t+1} - 1) + \epsilon \\
\frac{D_{t+1}}{W_t} = d(R_{p,t+1} - 1) + \epsilon.
\]

We report the results for the full regressions in panel A of table 1. However, a linear regression using the full data sample (including negative return observations and corresponding donations data) does not capture the non-linear nature of the donations processes used throughout our theoretical model. This can easily be shown as follows: To test for the appropriate way of revealing the true coefficient \(d\), we simulate normally distributed return observations and corresponding donations data according to equation 17 for a coefficient \(d\) equal to 0.2. Then we regress the simulated donations data on the randomly generated returns data to get an estimate of the coefficient \(d\). Using a simple linear regression (without intercept) on a simulated data sample yields an estimate of about 0.13. Hence the estimate is biased. Alternatively, to capture the nonlinear relation in equations (17) and (18) we condition the data set as follows: We use only positive observations of net returns and corresponding information on donations, that is \((D_{t+1}|R_{1,t+1} - 1 > 0)\) related to \((R_{1,t+1}|R_{1,t+1} - 1 > 0)\), and \((D_{t+1}|R_{p,t+1} - 1 > 0)\) related to \((R_{p,t+1}|R_{p,t+1} - 1 > 0)\). This conditioning enables us to investigate the relation of returns and donations for the positive part, leaving aside the negative part where donations are assumed to be zero. Applying this conditioning to the simulated data returns a coefficient of 0.2, which is the true coefficient of the donation generating process. Thus, we focus on the conditioned regressions for our empirical investigations, since it achieves an unbiased estimate.

The remaining four specifications are therefore based on the conditioning analysis, both for homogeneous and non-homogeneous cases. Panel B of table 1 provides the regression results for the conditioned regressions using the actual data. For all specifications we find positive and significant relations between returns and the donation rate, \(d\), at the 1% significance level. For the full data sample without conditioning, we find significantly lower values of the donation coefficient, \(d\) and much lower \(R^2\). For the conditioned estimation, the coefficients range from about 0.20 up to 0.24. The \(R^2\) values are remarkably large given that this is a pooled regression using all endowments, unlike our endowment specific theoretical model. This estimation provides strong empirical support for our assumptions about the relevant donations process.
<table>
<thead>
<tr>
<th>Panel A - full regressions</th>
<th>Panel B - conditioned regressions</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$\hat{d}$</td>
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**Table 1: Empirical Results** Panel A (the first 4 columns) show results for the regressions using the full data set, including negative returns. Panel B reports results for the data set conditioned on positive returns only. The first two columns of each panel correspond to the homogeneous linear specification. The last two columns of each panel correspond to the specifications without wealth homogeneity (with wealth effect). We implement the analyses as pooled regressions by aggregating data on all individual endowments which reported returns consistently over our sample from 2008 to 2013. Significance at the 1% confidence level is indicated by ***.

### 2.4 Simulations

Since the purpose of our model is to describe the optimal process of asset allocation and spending for an integrated donations process with time varying investment opportunities, we now employ the well-known model of Campbell and Viceira (1999) in a numerical optimization routine.

Campbell and Viceira (1999) assume that asset allocations are a linear function of the risk premium process, $x_t$ given by equation (6) and that the logarithm of the consumption to wealth ratio, $c_t - w_t$ is a quadratic function of $x_t$. This is based first on the homogeneity of the value function with respect to $W_t$, and also a log-linearization of the Euler conditions. That is

$$a_t = a_0 + a_1 x_t$$

and

$$c_t - w_t = b_0 + b_1 x_t + b_2 x_t^2.$$  (22)

where the coefficients $a_0, a_1, b_0, b_1, b_2$ are derived from the primary preference and return parameters through the method of undetermined coefficients. Although this is only an approximate solution method, it is fairly accurate for the vast majority of most $x_t$ values in the distribution. We compare this approximate solution method against a numerical optimization procedure in order to calibrate our model.

Using dynamic programming methods as discussed in Rust (1996), we optimize the following Bellman equation:

$$J(x_t)W_t = \max_{C_t, a_t} \left( (1 - \delta)C_t^{(1 - \rho) / \theta} + \delta \mathbb{E}((J(x_{t+1})W_{t+1})^{1 - \rho})^{\theta / (1 - \rho)} \right),$$

where $V(W_t, x_t) = J(x_t)W_t$. The optimization is performed subject to the accumulation equations (15) and (16) as well as the constraint that the endowment never goes below zero, subject to $W_t \geq 0 \ \forall t$. The value function $J(\cdot)$ is a fixed point to this functional Bellman equation and depends on a single
state variable, expected risk premia \( x_t \). In order to solve the Bellman equation (23) numerically we discretize over \( x_t \) using 50 equally spaced grid points for \( x_t \). Subsequently, the value function is fitted by using cubic spline approximations. We maximize the value function in each iteration over the control variables \( C_t/W_t \) and \( \alpha_t \) using a BFGS optimization (BFGS is a quasi-Newton method named after Broyden, Fletcher, Goldfarb and Shanno). In each iteration the spline parameters of the previous iteration are used to find the maximum of the value function, which is then used again to update the spline parameters. We calculate the norm between one iteration and the previous one, both in terms of the optimized level of the value function as well as of the optimal level of consumption, to judge whether convergence has been achieved. If the norm increases in any iteration from the previous iteration, the spline parameters are not updated, which increases the stability of the value function convergence. After reaching a sufficient level of convergence the value function can be applied to investigate optimal consumption and portfolio decisions for a given set of parameters \( \rho \), \( \psi \) and \( d \).

<table>
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<tr>
<th>value function</th>
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<th>return process</th>
<th>donations</th>
</tr>
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<td>( \rho ) ( \psi )</td>
<td>( \phi ) ( \mu ) ( \sigma^2_\eta )</td>
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Table 2: Parameters This table provides all relevant parameter values for the model. \( \rho \), \( \psi \) and \( d \) are varied, while all other parameters are fixed. The fixed parameter values are taken from the annual model in Campbell and Viceira (1996).

Table 2 provides the set of parameters used in our numerical optimization routine. Most of the parameter values remain fixed throughout the analysis. Risk aversion, the elasticity of intertemporal substitution and the relative donation inflows (\( d \)) are varied, though. The fixed parameters are taken from Campbell and Viceira (1996), where they are empirically estimated with 100 years of data. (In the empirical specification of Campbell and Viceira (1996), the price-to-dividend ratio served as a proxy for the expected risk premium.) Our relative donation parameters are close to our estimated values from the actual US endowment data.

Before we proceed to our main results, we first verified that our model replicates the essential results of Campbell and Viceira (1999) for a case with no donations. The formal results appear in Appendix B. Because the Campbell and Viceira (1999) model is only an approximation, we confirm that there is a small amount of nonlinearity in the asset allocation function. The slope of the optimal asset allocation function is slightly greater than the approximate solution. As far as the consumption-wealth ratio is concerned, here again there is a small amount discrepancy, as the approximate solution slightly understates consumption in relation to wealth for small values of \( x \) while overstating the precise solution for very high values. Nevertheless the differences are small in economic terms.

### 2.5 Optimal Policies for Positive Endowment Flows

We now illustrate the solution for positive donation inflows, according to equations (17) and (18). By contrast to the analytical results and the results assuming a linear-quadratic approximation, we now
find that the substitution effect is not uniform across all investment opportunities arising from the state variable, $x$.

Our numerical routine now relies on a simulation approach. We first generate a sample of 500 return paths with 50 observations each. As the expected return component of the risky asset depends on an autoregressive process ($x$) we remove the first 20 return observations for each path. With the remaining 30 return observations we then generate 500 wealth, consumption and donation paths which we now evaluate. Conditional moments for the return process are derived in Appendix C.

For the purposes of illustration consider the donation flows related to the net return on the risky asset. Figure 1 represents the solution to the asset allocation policies in this case. The Campbell-Viceira asset allocation in black is linear and lies everywhere above the exact solution to the positive donation problem which is in yellow. The substitution effect is evident for most values of $x$, since asset allocations are reduced compared to the case without donations inflows. Only at the lower endpoint where investment opportunities are poor and one does not expect any donations therefore do the solutions correspond. Figure 2 illustrates the form of the relation of optimal spending to wealth in this case. It is obvious that for very poor investment opportunities consumption increases relative to the endowment value, while for very good investment opportunities the spending rate is closer to the solution without donations. This can be understood when one considers that (due to the mean reverting risk premium process) when opportunities are bad, it is likely that investment returns will have been good. As a result, wealth is high and donations are positive so that a large fraction is spent out of the endowment. On the other hand when investment opportunities are good, it is likely that the endowment has just had a bad year. Then donations are nonexistent and spending rates converge to the Campbell-Viceira solution without donations.
Figure 1: Asset Allocation from Risky Asset Donations

This chart plots the optimal asset allocation given by numerical simulations using the assumption that relative risk aversion $\rho = 4$ and the elasticity of intertemporal substitution $\psi = 0.1$. Donations are assumed to be $d = 0.2$ and the other parameters are as in Table 2. The Campbell-Viceira solution without donations is represented in black while the yellow points represent the optimal asset allocation of the donation solution.
Next consider the impact of endogenous donations that depend on the excess return of the entire endowment portfolio as opposed to just the risky asset return. Figure 3 illustrates the solution. Now there is barely any substitution effect at all, except at the lower endpoint. There is still a possibility to get positive donation inflows because of the investment in the riskfree asset in this case. This is reflective of a positive feedback effect whereby investing more in the risky asset, especially when investment opportunities are good implies a higher probability of positive donation inflows. Figure 4 shows the corresponding consumption to wealth ratio for these optimal policies, again as a function of investment opportunities. As is apparent once more, the variation in consumption to wealth is significantly lower than it is without donations. In fact this relationship appears to be almost identical with the case where donations are related solely to the risky asset return. This results from the optimality of the asset allocation adjustments, as the future wealth distributions are almost identical in the two cases; hence consumption policies are similar.
Figure 3: Asset Allocation from Donations Related to Optimal Portfolio. This chart plots the optimal asset allocation given by numerical simulations using the assumption that relative risk aversion $\rho = 4$ and the elasticity of intertemporal substitution $\psi = 0.1$. Donations are assumed to be $d = 0.2$ and the other parameters are as in Table 2. The Campbell-Viceira solution without donations is represented in black while the yellow points represent the optimal asset allocation of the donation solution.
3 Endowment Size Effect

Having extensively considered the impact of endogenous donations when the system is homogeneous of degree 1 in endowment size, we now extend our model to a situation where size matters. We utilize the two donations specifications tested earlier: equations (19) and (20).

As with the homogeneous case, dynamic programming is used to solve for the optimal policy functions. Now the value function, \( V(x_t, W_t) \), has two dimensions. A two-dimensional grid is built over \( W_t \) and \( x_t \), discretized using 20 equally spaced points for \( x_t \) and 10 equally spaced points for \( W_t \). The value function is fitted by using two-dimensional (thin-plate) spline approximations. As before we iterate over the value function until essentially finding a fixed point in the functional space, at which point the optimal consumption and portfolio decisions can be computed. Once the solution is fixed, the results are then illustrated via Monte Carlo methods - sampling over random return and noise parameter processes. Parameters are as before in Table 2.

For the base case, Figure 5 illustrates the optimal allocation to risky assets and the log consumption-wealth ratio of the donation solution.
to-wealth ratio in terms of expected risk premia. $d$ is set to 0.2. The black lines in this figure represent the solutions for the Campbell and Viceira (1999) framework with additional donation inflows. This result is obtained by applying the optimal CV policy functions to simulated paths of the state variable $x$. We simulate 500 paths, each with 30 years of annual data for expected risk premia. Note that each period the university receives donation inflows according to equation (19). The CV solution assumes that the university does not anticipate these inflows and, hence, does not incorporate donations into his multi-period utility maximization. In contrast, the colored points depict optimal values for the control variables in our model setup. The colors are stratified by wealth levels. As wealth increases, the color goes from bright red towards dark blue. These points illustrate the numerically optimized portfolio and consumption decisions for an endowment that is aware of the underlying donation process and consequently takes anticipated donation into account for the utility maximization.

**Figure 5: Policy Rules from Size Dependent Risky Asset Donations** The left hand chart in this figure compares the optimal allocation to stocks ($\alpha$) in terms of risk premia ($x$) for the CV case without donations (black line) to the numerically optimized values of our model with endogenous (anticipated) donation inflows. The right hand chart in this figure provides a similar comparison for the log consumption-to-wealth ratio. The donation process assumed for our model framework is given by equation (19) and the set of parameter values is $\{\rho, \psi, d\} = \{4, 0.1, 0.2\}$. Red points represent low values of endowment size, while blue points depict high values of endowment size.

The results in Figure 5 are based on the donation process in equation (19), thus donations are procyclical with respect to capital market returns. Both, positive endowment returns and positive donations lead to higher endowment wealth and enable higher levels of consumption. Since returns and donations are positively correlated in our model, universities implicitly have higher exposure to risky assets. As a consequence, endowments allocate less to risky assets than they would if they didn’t anticipate the underlying process for donation inflows. Table 3 shows that the magnitude of this substitution effect is about 13 percentage points on average. However, $\alpha$ is not linear in expected risk premia in our model, as can be seen from the left graph in Figure 5. As with the wealth-independent case, we observe a substitution effect for most values of $x$. However, we observe an important deviation at the
top end, i.e., when investment opportunities are good. Now we actually observe high allocations to the risky asset. The reason for this is that donations are always non-negative in this form of the donations process. Thus, donations can be interpreted as an option for the university. If returns turn out to be positive high donation inflows amplify the utility gain for the university, while they do not exacerbate losses if returns are indeed negative. The cost of this “option” is a greater loss in unlikely adverse scenarios as compared to an agent allocating assets according to the CV solution (black line). If risk premia are high enough, the probability of such adverse scenarios becomes marginally low and so the “option” gets very cheap so that rational endowments allocate substantially more into stocks than agents following the CV solution.

<table>
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<th>mean</th>
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</table>

Table 3: Functional Form of Policy Rules

As mentioned previously the empirical literature on university endowments has established important regularities in terms of endowment size. Therefore we analyze the results separately by size of the endowment. At the upper endpoint, smaller endowments are more aggressive with their long positions; at the lower end larger endowments are more aggressive in the sense of shorting the risky asset to a greater extent. Hence, small endowments try to exploit very good investment opportunities to accumulate wealth more quickly.

It is not surprising that the spending rate (the log consumption-to-wealth ratio depicted in the right hand graph of Figure 5) is higher in our setup than in the CV world where donation inflows are unanticipated and the underlying donation process is not revealed to the representative agent. For larger endowments (depicted in blue), the log consumption-to-wealth ratio has less curvature with respect to $x$ than in the CV case. However for small endowments there is greater variation. When investment opportunities are great small endowments spend relatively more than large endowments. This can be justified by a substantially higher allocation to the risky asset in this case. Since the difference in the allocation between small and large endowments is greater than the difference in the spending rate, small endowments accumulate wealth at a faster rate but at the same time reduce the gap in dollar spending relative to large endowments.

Figure 6 illustrates how the consumption and portfolio decisions change if we assume the donation process to evolve as in equation (20). Now donations are positively correlated with endowment returns, which enables universities to receive donations even if market returns are negative. Obviously this
is only feasible if the endowment appropriately chooses $\alpha$ to be negative. The average allocation to risky assets is much closer to the CV solution as donations do expose universities less to the return on risky assets. For medium-range values of $x$ the optimal allocation to stocks is typically between 0 and 1. For this reason there are many occasions where market returns are slightly negative but due to the allocation to the risk-free asset (which yield the risk-free rate) the endowment return is still positive and, thus, donations are positive as well. Hence, for those medium-range values of $x$ the total expectation of returns plus donations is often positive, providing utility gains for the university. As a consequence, the optimal allocation to stocks is higher than in Figure 5 and similar to the CV solution. If investment opportunities are bad the allocation to stocks decreases and might turn negative. However, big short positions lose money if market returns turn out to be positive. Due to the relatively large allocation to risk-free assets that results from being short stocks there are again cases where endowment returns are slightly positive despite incorrectly anticipating negative market returns, and so donations are positive again. If short positions are too extreme and the market returns are unexpectedly positive, endowment returns will ultimately have negative realizations. In this case donations will be zero and utility will be low. This is why endowments are more conservative in shorting the market in our model setup than in the CV setup. For large expected risk premia, the same rationale applies as in Figure 5, since positive donations are very likely in both cases.

The size effect for high values of $x$ is comparable to the case when donations depend on the risky asset return. However, for medium-range values of $x$ large endowments tend to allocate systematically more to the risky asset than small endowments. Furthermore, the allocation of small endowments is more convex at the lower end in this case, thus small endowments are less aggressive in shorting the market.

The log consumption-to-wealth ratio is again more convex, as a function of $x$, for small endowments than for large endowments. This is even more the case than for risky asset donations. The finding that small endowments try to accumulate wealth more quickly in good states of the world is still true if donations are endogenous in endowment returns. However, spending rates of small endowments appear systematically lower even for low and medium-range values of $x$. 
Figure 6: Policy Rules from Size Dependent Donations related to Optimal Portfolio. The left hand chart in this figure compares the optimal allocation to stocks ($\alpha$) in terms of risk premia ($x$) for the CV case where donations are expected (black line) to the numerically optimized values of our setup with endogenous (anticipated) donation inflows. The right hand chart in this figure provides a similar comparison for the log consumption-to-wealth ratio. The donation process assumed for the alternative model framework is given by equation (20) and the set of parameter values is $\{\rho, \psi, d\} = \{4, 0.1, 0.2\}$. Red points represent low values of endowment size, while blue points depict high values of endowment size.

A comparison of summary statistics is provided in Table 3. We see here that for both donations processes, the asset allocation to risky assets is lower with positive donations, at the median value of $x$. However consumption to wealth is significantly higher. For extreme low investment opportunities, the asset allocation remains below (short) the CV solution for the case with donations related to risky returns, while it is less short for the portfolio donations process. For high values of $x$, both donations processes produce high risky asset allocations compared to CV and also very high spending levels in comparison, with the effect being greater in the case with donations related to overall portfolio rates of return.

To further illustrate the effect of endowment size it is interesting to plot the log consumption-to-wealth ratio as a function of wealth. The result is seen in Figure 7. The left hand graphs represent the CV solution, while the right hand graphs provide the results of our model. The top panel assumes donations to depend on the return of the risky asset, whereas the bottom panel assumes donations to be procyclical with respect to endowment returns. Figure 7 reveals that the CV solution yields a minimum spending rate which is independent of wealth. By contrast, this minimum spending rate increases in wealth in our framework. This is especially true for the donation process that is related to endowment returns. The figure suggests that very small endowments tend to sacrifice consumption in order to grow endowment assets faster and achieve higher consumption in absolute terms later. For medium to large-sized endowments the minimum spending rate is more or less constant again. The reason for the greater dependence on the level of wealth in the bottom panel might be that donations can be increased through superior asset management even in adverse capital market scenarios. That
Figure 7: Spending in terms of Wealth. This figure illustrates optimal log consumption-to-wealth ratios in terms of the log of wealth. The left hand graphs provide results for the CV case, while the right hand graphs show the corresponding results for our model setup. The top panel assumes donations to depend on the return of the risky asset, whereas the bottom panel assumes donations to depend on the endowment return. Results are plotted for parameter values \(\{\rho, \psi, d\} = \{4, 0.1, 0.2\}\). Red points represent low values of endowment size, while blue points depict high values of endowment size.
is, given that the asset allocation is chosen appropriately, there are more scenarios where donation inflows are positive if the donation process depends on the endowment return. Thus, it is even more attractive to sacrifice current consumption to grow assets faster as the fraction of donations \( d \) will be applied to a larger wealth level later due to more accumulated donations. This in turn enables much higher dollar levels of spending in the future.

To summarize our findings on the wealth effect larger endowments have a larger exposure in the risky asset when donations depend on endowment returns. This confirms earlier findings in the literature. However, for exceptionally good investment opportunities small endowments tend to allocate more into the risky asset. Moreover, small endowments spend relatively less as a fraction of wealth in most states of the world. These two effects imply that small endowments try to accumulate wealth more quickly than large endowments.

Analyzing the log consumption-to-wealth ratio as a function of log wealth for various sets of parameters, we find that the wealth dependence increases with higher elasticity of intertemporal substitution. In contrast, wealth dependence of optimal spending rates tends to decrease with higher risk aversion. If \( \psi \) increases above one the minimum spending rate shown in Figure 7 turns to a wealth dependent maximum spending rate. These findings can be interpreted as follows: The more willing an endowment is to shift consumption through time depending on the prevailing investment opportunities and the more willing to tolerate risk the endowment is, the more wealth dependent is the optimal spending rate.

Finally, Tables 4 and 5 provide summary statistics for various levels of \( \rho \) and \( \psi \). Consistent with Campbell and Viceira (1999) we find that on average the optimal allocation to stocks is inversely related to risk aversion. Varying \( \psi \) has only minor effects on the amount allocated to risky assets, \( \psi \) and \( \alpha \) tend to be negatively related, though. The curvature of \( \alpha \) as a function of \( x_t \) is negatively related to the level of risk aversion, as can be seen from the spread between the first quartile and the third quartile in Table 4. That means that the higher the risk aversion the less aggressive the endowment gets in its allocation decision. Thus, the divergence from the original CV results due to the introduction of endogenous donations is especially large for endowments with low risk aversion. Moreover, comparing the top and the bottom panel of Table 4 reveals that average allocations to stocks are higher if donations depend on endowment returns instead of risky asset returns. This confirms the findings on the substitution effect elaborated on above.

For the log consumption-to-wealth ratio the influence of \( \psi \) is more pronounced. Average spending rates tend to be negatively related to the level of intertemporal elasticity of substitution if \( \psi \) is below one, which seems to be a plausible assumption for endowments. The curvature of optimal spending rates as a function of \( x_t \) also depends on the level of the intertemporal elasticity of substitution. The curvature of this function decreases in \( \psi \), which means that spending rates are more sticky with respect to expected risk premia if the intertemporal elasticity of substitution is higher. Note, however, that high expected risk premia usually follow drawdowns in the endowment value. Thus, sticky spending rates correspond to highly fluctuating absolute consumption. For this reason dollar consumption is more sticky for low values of \( \psi \). For values above one, the function flips sign and decreases in \( x_t \), again
consistent with CV. Furthermore, average spending rates are negatively related to the degree of risk aversion.

4 Tests

This section employs actual time series and cross-sectional data on endowment returns, risky asset returns, endowment spending, asset allocation and donations to investigate the empirical predictions of our integrated model. The university endowment data are as before from NACUBO/Commonfund. The data are from the time period 2006 to 2013, however, for those models where we need dollar levels of spending or donations we use data from 2009 to 2013 since these data series are unavailable beforehand.

4.1 Investment Opportunities

Our theory predicts that the allocation to risky assets should be monotonically increasing in a state variable representing investment opportunities. Although this is not our main hypothesis, it does represent a basic diagnostic test to see whether our data and the way in which it is aggregated is a reasonable

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<td>8 1.33</td>
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Table 4: Functional Form of $\alpha$ for various Parameters This table provides summary statistics for the optimal allocation to stocks in terms of expected risk premia $x_t$ for various sets of parameters. The top panel assumes donations to depend on the return of the risky asset, while the bottom panel assumes donations to be positively related to endowment returns.
Table 5: Functional Form of the Log Consumption-to-Wealth Ratio for various Parameters

This table provides summary statistics for the optimal log consumption-to-wealth ratio in terms of expected risk premia $x_t$ for various sets of parameters. The top panel assumes donations to depend on the return of the risky asset, while the bottom panel assumes donations to be positively related to endowment returns.

The implementation of the model. Therefore we estimate a regression in the following linear specification:

$$a_{it} = a_{i0} + a_1 x_{it} + \epsilon_{it},$$

where investment opportunities, $x_{it}$, are possibly specific to endowment $i$ at time $t$.

Recall that asset allocations are delineated in 12 different categories. Since our model features only a single risky asset, we aggregate the actual allocations to domestic equities, international equities and private equity as well as fixed income to calculate the fraction of assets that are invested in risky assets for each endowment in each year. We also need a measure of investment opportunities. For each endowment, $i$, we compute the time series average of asset allocations in each category. We define this to be the “strategic policy” weights. We use these policy weights to compute a weighted average dividend yield plus share repurchase yield for the domestic equity class, the S&P 500, the dividend and repurchase yields for international equities as proxied by the Eurostoxx, Nikkei and FTSE. Investment opportunities in the case of fixed income are represented as the spread between Baa and Aaa rated corporate bonds. We then use the policy weights times the actual rates of return in these asset categories to derive an endowment specific return on the risky part of the portfolio.

Model 1 in Table 6 shows the slope coefficient of a fixed effects (with respect to endowment) panel
regression. The coefficient is positive and statistically significant. Model 2 deletes the fixed income component and therefore only measures investment opportunities as the weighted average of the equity class returns. We find here that the coefficient is still positive and significant and the $R^2$ is greater. Model 3 measures investment opportunities as the difference between the equity related weighted average and the fixed income average. We find that the relationship remains significant and positive. Finally, in model 4, we measure the LHS variable in the regression as the difference between the actual equity-related asset allocations and the fixed income allocations. This model embodies an assumption therefore that the alternative asset allocation $1 - a_{it}$ is a combination of both cash and fixed income. Again the coefficient is statistically significantly positive. Therefore we believe on the basis of these tests that asset allocation weights vary with respect to investment opportunities in a dynamic way that is consistent with our measures of investment opportunities.

<table>
<thead>
<tr>
<th></th>
<th>(1) risky fraction</th>
<th>(2) risky fraction</th>
<th>(3) stocks minus bonds</th>
<th>(4) stocks minus bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>inv. opp.</td>
<td>2.1386***</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(6.0928)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>inv. opp. (excl. fixed income)</td>
<td></td>
<td>5.1382***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(11.6690)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>diff. in inv. opp. stocks vs. bonds</td>
<td></td>
<td></td>
<td>6.0807***</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(6.8454)</td>
<td></td>
</tr>
<tr>
<td>inv. opp. (excl. fixed income)</td>
<td></td>
<td></td>
<td></td>
<td>6.5234***</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(9.0504)</td>
</tr>
<tr>
<td>No. Obs.</td>
<td>2110</td>
<td>2110</td>
<td>2108</td>
<td>2108</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.0151</td>
<td>0.0606</td>
<td>0.0218</td>
<td>0.0374</td>
</tr>
</tbody>
</table>

Table 6: Impact of Investment Opportunities This table illustrates the effect of changes in investment opportunities on asset allocation. Model (1) regresses the fraction of risky assets of each and endowment and each year on endowment/year specific investment opportunities by means of an endowment fixed effect panel regression. Endowment specific investment opportunities are modeled by a weighted average of asset class specific investment opportunities. As weights we use the time series average of observed endowment specific asset class weights (policy portfolios). Investment opportunities for domestic equities are defined as dividend yield plus repurchase yield of the S&P 500, whereas for international equities we use dividend yields and repurchase yields of the Eurostoxx, the Nikkei and the FTSE. For fixed income we use the spread between yields on Baa and Aaa rated bonds. Model (2) regresses the risky asset fraction on investment opportunities excluding the credit spread. Model (3) regresses the difference of the equity allocation and the fixed income allocation on the difference of investment opportunities of equities and fixed income. Model (4) regresses the difference of the equity allocation and the fixed income allocation on investment opportunities for stocks only. T-stats are presented in parantheses below the coefficients.

4.2 Substitution Effect

Our main predictions concerning the substitution effect of donations are that for endowments with a greater link between returns and donations, there should be a lower allocation to risky assets. Furthermore, when the feedback effect is measured with respect to endowment returns, as compared to asset returns, the substitution effect is less. As with the previous test we adopt several models of the incidence of substitution to handle robustness. The form of the linear regression test is as follows:

$$a_{it} = a_0 + a_1 d_i + \epsilon_{it},$$

where $d_i$ is a donation variable representing the relation between donations and returns such as in equation (8), equation (9), equation (17) and equation (18). The results of our linear estimation are
reflected in Table 7. This table contains two panels. On the right hand side the substitution effect is measured with respect to risky asset returns as modeled in the previous subsection. On the left hand panel, endowment returns are utilized. Model 1 defines \( d_i \) as a correlation between yearly changes in donation rates and endowment returns using the available time series data. Model 2 defines \( d_i \) as in our donations test from a time series regression of each endowment's donations on the endowment returns, retaining only positive returns data. Model 3 utilizes all returns data, thus implying a linear donations process. In these specifications we find strong support for our nonlinear donations model, i.e., model 2, in which donations are positively related to good years for the endowment. The other models do not perform well, and in fact the coefficients are of the wrong sign for the substitution effect to hold.

Models 4, 5 and 6 in Table 7 contain the corresponding regressions where the \( d_i \) variables are determined from either the correlation coefficient (model 4), or else from linear regressions on risky asset returns. We get basically the same results; there is a significant substitution effect. It is worthwhile noting that the magnitude of the substitution effect is only about half that of the model with endowment returns. This result is opposite to that predicted by our theoretical model in which the substitution effect is lower for donations related to endowment returns instead of its constituents. One possibility for this contrary result is that we have error in our measure for endowment risky asset returns while the actual endowment return is measured exactly. This can bias downward the substitution effect.

<table>
<thead>
<tr>
<th></th>
<th>endowment returns</th>
<th>risky asset returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>intercept</td>
<td>0.9237***</td>
<td>0.9398***</td>
</tr>
<tr>
<td></td>
<td>(160.9)</td>
<td>(142.2)</td>
</tr>
<tr>
<td>donation correlation</td>
<td>0.0032</td>
<td>0.0113</td>
</tr>
<tr>
<td></td>
<td>(0.2870)</td>
<td></td>
</tr>
<tr>
<td>model based don. risk</td>
<td>-0.0517***</td>
<td>-0.0328**</td>
</tr>
<tr>
<td></td>
<td>(-2.9550)</td>
<td>(-1.9690)</td>
</tr>
<tr>
<td>don. risk without filtering</td>
<td>-0.0176</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-0.6640)</td>
<td></td>
</tr>
<tr>
<td>No. Obs.</td>
<td>244</td>
<td>304</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.0003</td>
<td>0.0218</td>
</tr>
</tbody>
</table>

Table 7: Substitution Effect. This table illustrates the effect of donation risk on asset allocation. Model (1) uses the correlation of year on year changes in donations with returns as a proxy for the donation risk of each endowment. Model (2) uses the coefficients of regressions of donation fractions on returns for all endowments but using data in years of positive returns only, consistent with our theory. Model (3) does the same as Model (2) but without filtering the returns. In all models the fraction of risky assets is regressed on an intercept and the respective measure of donation risk. Models (4), (5) and (6) are the analogous results for the risky asset returns. The analysis is set up as a stacked regression using all available endowments. T-stats are presented in parantheses below the coefficients.

4.3 Size Effect

We now investigate the size effect empirically with respect to asset allocations and spending. Based on the simulations in section 3, we infer that asset allocations should be positively impacted by size for
most of the relevant range in the investment opportunity variables. Results are presented in Table 8.

Model 1 therefore contains the regression results of an estimation of

$$a_{it} = a_0 + a_1 \log W_{it} + \epsilon_{it},$$

while the spending estimations are given as in

$$c_{it} = a_0 + a_1 \log W_{it} + \epsilon_{it}.$$  

We see that indeed asset allocations to risky assets are greater for larger endowments.

As far as spending is concerned, we find that the amount and rate of spending should be greater for larger endowments, at least over the middle range of investment opportunities, where most outcomes are found. Model 2 uses the logarithm of spending amounts, while model 3 uses the logarithm of spending rates. Both models 2 and model 3 feature a positive and significant size impact on spending, as predicted by the theory.

<table>
<thead>
<tr>
<th></th>
<th>risky asset fraction</th>
<th>log spending</th>
<th>log cons./wealth ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>0.8576***</td>
<td>-4.5962***</td>
<td>-4.6840***</td>
</tr>
<tr>
<td></td>
<td>(31.938)</td>
<td>(-17.787)</td>
<td>(-19.162)</td>
</tr>
<tr>
<td>log size</td>
<td>0.0041***</td>
<td>1.0689***</td>
<td>0.0746***</td>
</tr>
<tr>
<td></td>
<td>(2.9525)</td>
<td>(79.765)</td>
<td>(5.8596)</td>
</tr>
<tr>
<td>No. Obs.</td>
<td>2412</td>
<td>1364</td>
<td>1364</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.0042</td>
<td>0.8348</td>
<td>0.0257</td>
</tr>
</tbody>
</table>

Table 8: Size Effect

This table illustrates the effect of size on asset allocation and spending. Model (1) regresses the fraction of risky assets of each endowment and each year on log endowment size and an intercept. Model (2) regresses the log of spending on log endowment size and an intercept. Model (3) regresses the log consumption to wealth ratio on log size and an intercept. The analysis is implemented by means of a stacked regression using all available endowments. T-stats are presented in parantheses below the coefficients.

4.4 Multivariate Analysis

We have found support for the theoretical model in our previous three subsections using univariate analysis. We now subject the theory to a stricter test by using multiple influences of variables of interest to our theory on asset allocation and spending. Our results are presented in Tables 9 and 10 using various specifications of fixed effects panel regressions. The left panel features endowment returns while the right panel features risky asset returns. Models (1), (2), (5) and (6) relate endowment asset allocation (the fraction of risky assets) to several regressors and their interactions. Models (3), (4), (7) and (8) use the log consumption to wealth ratio (the spending rate) as an additional dependent variable.

There is considerable robustness in our tests of the dependence on asset allocations to various models in Table 9. The size effect almost always shows up very positively and statistically significant. Investment opportunities are also positive and significant in some of the specifications. However the interaction between investment opportunities and size is negative. This implies that there is less varia-
tion over the investment opportunities cycle for large firms as compared to small firms. This is consistent with our theoretical analysis. We also find that the interactive effect between spending rates and investment opportunities is negative. This implies that endowments with high payout rates have lower elasticities in varying their risky asset exposures. What is notable is that the donation variable seems to be less important than many of the others. Of course this might be due to the fact that donations influences other variables in the model and the effects are picked up better by these.

Table 10 reverses the regression framework, where now asset allocation and other variables are used to explain spending. As before, we find that size is the most reliable indicator of spending rates and that the impact is negative, even controlling for all other effects we can observe. Recall that in our theory investment opportunities exhibited a nonlinear relation to spending. We find that the linear part of the regression on investment opportunities is insignificant as expected; however the nonlinear part is concave, rather than convex. This has the implication that the elasticity of intertemporal substitution is greater than one. We also find some qualified support for an interaction between asset allocation and investment opportunities such that spending rates decrease with higher asset allocations and better investment opportunities. We also get a positive relation between donation risk and good investment opportunities and the spending rates of university endowments. In other words, universities maintain their spending even with good investment opportunities for substantial donation levels.

5 Illustration

In order to illustrate our proposed method of integrating optimal asset allocation and spending into a model with endogenous donations, we implement a backtest of our model over the period from 1972-2012 for a representative university endowment. In keeping with the simulated model, we use the VAR model estimates from Campbell and Viceira (1996), which were estimated over a much longer historical period which terminated in 1993. We use annual total returns on a value weighted equity index from CRSP as our risky asset. The risk free rate is represented by 12 months treasury bills, which is obtained from the board of governors of the federal reserve system. To model the state variable \( x \) representing investment opportunities we use the logarithm of the 12 month trailing dividend yield from CRSP and enhance it by a 12 month trailing log share repurchase yield (total share repurchases over the last 12 months divided by the year-end market capitalization of the equity index). Share repurchases are included, since they contribute substantially to returns to shareholders since the mid-eighties; see for instance Fama and French (2001). Aggregate share repurchase data from 1972 to 2000 comes from Grullon and Michaely (2002), who base their estimates on Compustat data; we extend it to 2012 using Bloomberg data for S&P 500 repurchases as a proxy. Because we do not have direct data on donations, we use equations (19) and (20) setting \( \epsilon_t = 0 \).

To backtest our model we evaluate the value function for \( \rho = 4 \) and \( \psi = 0.1 \) using the state variable just described. This provides us with the optimal allocation to risky assets and optimal spending.

6 Although there is about a 20 year overlap, our purpose here is for illustrative purposes rather than conducting an empirical test.
Table 9: Asset Allocation - Combined Effects This table summarizes the results of several regressions explaining asset allocation (the fraction of risky assets is the left hand side variable). Model (1) relates the fraction of risky assets for endowments to investment opportunities, endowment size, interaction terms using both of them and interaction terms including donation risk. Model (2) uses only a subset of regressors. Models (5) and (6) repeat the same exercise but using risky asset returns instead of endowment returns where appropriate. Models (3), (4), (7) and (8) also include the log consumption to wealth ratio as a regressor. All Models are estimated as panel regressions using endowment fixed effects. T-stats are presented in parantheses below the coefficients.

<table>
<thead>
<tr>
<th></th>
<th>endowment returns</th>
<th></th>
<th>risky asset returns</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>investment opportunities</td>
<td>12.1722</td>
<td>16.3811**</td>
<td>7.9315</td>
<td>10.7290</td>
</tr>
<tr>
<td></td>
<td>(1.4514)</td>
<td>(2.1041)</td>
<td>(0.7857)</td>
<td>(1.0852)</td>
</tr>
<tr>
<td>log size</td>
<td>0.0656**</td>
<td>0.0905***</td>
<td>0.0638**</td>
<td>0.0893***</td>
</tr>
<tr>
<td></td>
<td>(2.5489)</td>
<td>(3.9843)</td>
<td>(2.0543)</td>
<td>(3.0969)</td>
</tr>
<tr>
<td>log consumption to wealth (lcw)</td>
<td></td>
<td>0.0936</td>
<td>0.1211</td>
<td>0.1374</td>
</tr>
<tr>
<td></td>
<td>(0.9278)</td>
<td>(1.2560)</td>
<td>(1.4336)</td>
<td>(1.2560)</td>
</tr>
<tr>
<td>invest. opport. x log size</td>
<td>-0.6133</td>
<td>-0.8247**</td>
<td>-0.6261</td>
<td>-0.8349*</td>
</tr>
<tr>
<td></td>
<td>(-1.4014)</td>
<td>(-2.0162)</td>
<td>(-1.3114)</td>
<td>(-1.8020)</td>
</tr>
<tr>
<td>invest. oppor.t x donation risk</td>
<td>0.5360</td>
<td>1.7383</td>
<td>0.0695</td>
<td>1.0433</td>
</tr>
<tr>
<td></td>
<td>(0.1392)</td>
<td>(0.3999)</td>
<td>(0.0173)</td>
<td>(0.2425)</td>
</tr>
<tr>
<td>log size x donation risk</td>
<td>0.0968*</td>
<td>0.0874</td>
<td>0.0910</td>
<td>0.0813</td>
</tr>
<tr>
<td></td>
<td>(1.6823)</td>
<td>(1.3978)</td>
<td>(1.5881)</td>
<td>(1.3157)</td>
</tr>
<tr>
<td>lcw x invest. oppor.</td>
<td>-1.3597</td>
<td>-1.7603*</td>
<td>-1.8732**</td>
<td>-1.7603*</td>
</tr>
<tr>
<td></td>
<td>(-1.4070)</td>
<td>(-1.8411)</td>
<td>(-1.9960)</td>
<td>(-1.8411)</td>
</tr>
<tr>
<td>lcw x log size</td>
<td>-0.0020</td>
<td>-0.0025</td>
<td>-0.0031</td>
<td>-0.0025</td>
</tr>
<tr>
<td></td>
<td>(-0.5119)</td>
<td>(-0.6755)</td>
<td>(-0.8423)</td>
<td>(-0.6755)</td>
</tr>
</tbody>
</table>

|                        | (5)               | (6)                   | (7)                 | (8)                   |
| No. Obs.              | 1136              | 1206                  | 1048                | 1144                  |
| $R^2$                 | 0.0236            | 0.0232                | 0.0324              | 0.0257                |

Table 9: Asset Allocation - Combined Effects This table summarizes the results of several regressions explaining asset allocation (the fraction of risky assets is the left hand side variable). Model (1) relates the fraction of risky assets for endowments to investment opportunities, endowment size, interaction terms using both of them and interaction terms including donation risk. Model (2) uses only a subset of regressors. Models (5) and (6) repeat the same exercise but using risky asset returns instead of endowment returns where appropriate. Models (3), (4), (7) and (8) also include the log consumption to wealth ratio as a regressor. All Models are estimated as panel regressions using endowment fixed effects. T-stats are presented in parantheses below the coefficients.
<table>
<thead>
<tr>
<th></th>
<th>endowment returns</th>
<th></th>
<th>risky asset returns</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td></td>
<td>(-0.4383)</td>
<td>(-0.2133)</td>
<td>(0.1374)</td>
<td>(0.4952)</td>
</tr>
<tr>
<td>investment opportunities$^2$</td>
<td>-3526.7618***</td>
<td>-3647.8200***</td>
<td>-4103.6000***</td>
<td>-3647.8199***</td>
</tr>
<tr>
<td></td>
<td>(-3.6170)</td>
<td>(-3.7622)</td>
<td>(-4.1587)</td>
<td>(-3.7622)</td>
</tr>
<tr>
<td>log size</td>
<td>-0.8172***</td>
<td>-0.7686***</td>
<td>-0.9325***</td>
<td>-0.8328***</td>
</tr>
<tr>
<td></td>
<td>(-4.7410)</td>
<td>(-4.8949)</td>
<td>(-4.3734)</td>
<td>(-4.4625)</td>
</tr>
<tr>
<td>asset allocation</td>
<td></td>
<td></td>
<td>0.6934</td>
<td>2.3581</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.2309)</td>
<td>(0.8195)</td>
</tr>
<tr>
<td>invest. opport. x log size</td>
<td>1.3411</td>
<td>0.8320</td>
<td>2.5082</td>
<td>2.1824</td>
</tr>
<tr>
<td></td>
<td>(0.4554)</td>
<td>(0.2947)</td>
<td>(0.8543)</td>
<td>(0.7769)</td>
</tr>
<tr>
<td>invest. opport. x donation risk</td>
<td>43.7150</td>
<td>32.3312</td>
<td>68.406**</td>
<td>51.2385*</td>
</tr>
<tr>
<td></td>
<td>(1.6214)</td>
<td>(1.1968)</td>
<td>(2.5288)</td>
<td>(1.8897)</td>
</tr>
<tr>
<td>log size x donation risk</td>
<td>0.2707</td>
<td>0.3683</td>
<td>6.5212</td>
<td>0.1861</td>
</tr>
<tr>
<td></td>
<td>(0.6998)</td>
<td>(0.9324)</td>
<td>(0.1684)</td>
<td>(0.4705)</td>
</tr>
<tr>
<td>asset allocation x invest. opport.</td>
<td>-64.0553</td>
<td>-79.9220**</td>
<td>-73.0721*</td>
<td>-79.9220**</td>
</tr>
<tr>
<td></td>
<td>(-1.5911)</td>
<td>(-1.9743)</td>
<td>(-1.7867)</td>
<td>(-1.9743)</td>
</tr>
<tr>
<td>asset allocation x log size</td>
<td>0.1067</td>
<td>0.10670</td>
<td>0.1120</td>
<td>0.0589</td>
</tr>
<tr>
<td></td>
<td>(0.8553)</td>
<td>(0.5203)</td>
<td>(0.9093)</td>
<td>(0.5203)</td>
</tr>
<tr>
<td>No. Obs.</td>
<td>1052</td>
<td>1074</td>
<td>1048</td>
<td>1070</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.0716</td>
<td>0.0709</td>
<td>0.0649</td>
<td>0.0641</td>
</tr>
</tbody>
</table>

Table 10: Log Consumption to Wealth Ratio - Combined Effects This table summarizes the results of several regressions explaining the spending rate (the log consumption to wealth ratio is the left hand side variable). Models (1), (2), (5) and (6) relate the log consumption to wealth ratio of our sample of endowments to a set of regressors. Models (3), (4), (7) and (8) also include the asset allocation (an interaction terms thereof) as an regressor. All Models are estimated as panel regressions using endowment fixed effects. T-stats are presented in parentheses below the coefficients.
We allocate funds accordingly and observe returns. The black solid line in Figures 8 and 10 illustrates the resulting wealth accumulation in the presence of endogenous donations. The first figure assumes donations depend on risky asset returns, whereas the latter assumes donations depend on total endowment returns. We compare this wealth accumulation with three heuristic based consumption portfolio rules. All of them use a static 60:40 allocation to risky assets and treasury bills. The Merton rule (red line) applies a constant target spending rate to current wealth, while the moving average rule (blue line) applies this target spending rate to a three year moving average of wealth. The hybrid rule (green line) spends 30% of last years spending plus 70% of the target spending rate applied to current wealth. For comparability in the case of all these heuristics the target spending rate is chosen to be 6.8%, which corresponds to the average spending rate of our optimal model.

Figure 8: Backtest - Wealth Accumulation for Risky Asset Donations. This figure illustrates the backtested wealth accumulation using the asset allocation and spending according to our model optimum (black solid line) and compares it to heuristic based rules. We use our base parameter set for the optimal allocation. All heuristic based rules implement a static 60% allocation to risky assets and a target spending rate of 6.8%. The heuristic based rules are a Merton type rule, a moving average rule and a hybrid spending rule. The backtest period is 1972 to 2012. We use total returns on a value weighted index from CRSP and the 12 month treasury bill rate as the risk free rate. Investment opportunities are approximated by the log dividend yield on the CRSP index plus a share repurchase yield. For all lines plotted in this chart we assume donations to depend on risky asset returns.

Starting at a normalized wealth level of 1.0, Figure 8 shows that the optimized rule finishes with the highest wealth accumulation over time. There are only minor differences between the heuristic based
rules. The mean allocation to risky assets in the optimized model over the backtest period is 59% with a standard deviation of 56%, which highlights the need for dynamic asset allocation and spending rules.

Figure 9: Backtest - Spending Levels for Risky Asset Donations

This figure illustrates the backtested spending levels using the asset allocation and spending according to our model optimum (black solid line) and compares it to heuristic based rules. We use our base parameter set for the optimal allocation. All heuristic based rules implement a static 60% allocation to risky assets and a target spending rate of 6.8%. The heuristic based rules are a Merton type rule, a moving average rule and a hybrid spending rule. The backtest period is 1972 to 2012. We use total returns on a value weighted index from CRSP and the 12 month treasury bill rate as the risk free rate. Investment opportunities are approximated by the log dividend yield on the CRSP index plus a share repurchase yield. For all lines plotted in this chart we assume donations to depend on risky asset returns.

Figure 9 shows the corresponding spending levels over time. Importantly, our model achieves higher absolute spending levels than all of the heuristic based rules for virtually every time period.
Figure 10: Backtest - Wealth Accumulation for Donations related to Optimal Portfolio. This figure illustrates the backtested wealth accumulation using the asset allocation and spending according to our model optimum (black solid line) and compares it to heuristic based rules. We use our base parameter set for the optimal allocation. All heuristic based rules implement a static 60% allocation to risky assets and a target spending rate of 6.8%. The heuristic based rules are a Merton type rule, a moving average rule and a hybrid spending rule. The backtest period is 1972 to 2012. We use total returns on a value weighted index from CRSP and the 12 month treasury bill rate as the risk free rate. Investment opportunities are approximated by the log dividend yield on the CRSP index plus a share repurchase yield. For all lines plotted in this chart we assume donations to depend on endowment returns.

Figures 10 and 11 present backtested wealth and spending levels for the case where donations are endogenously determined by endowment returns instead of risk asset returns. The results remain qualitatively the same. The mean allocation to risky assets is 70% in this case with a standard deviation of 56%.
Figure 11: Backtest - Spending Levels for Donations related to Optimal Portfolio This figure illustrates the backtested spending levels using the asset allocation and spending according to our model optimum (black solid line) and compares it to heuristic based rules. We use our base parameter set for the optimal allocation. All heuristic based rules implement a static 60% allocation to risky assets and a target spending rate of 6.8%. The heuristic based rules are a Merton type rule, a moving average rule and a hybrid spending rule. The backtest period is 1972 to 2012. We use total returns on a value weighted index from CRSP and the 12 month treasury bill rate as the risk free rate. Investment opportunities are approximated by the log dividend yield on the CRSP index plus a share repurchase yield. For all lines plotted in this chart we assume donations to depend on endowment returns.

6 Conclusions

It is well known that while average university endowments in the US are characterized by an average 60:40 percent equity to bonds ratio and about 5% spending per year, there is significant cross-sectional heterogeneity in asset allocation policy and strategic weights, and significant time-series variation in spending policies as well. We have developed a new model for university endowments, building on the standard intertemporal consumption-portfolio problem by adding endogenous donation inflows. The purpose of this research is to study the impact on optimal university asset allocations, spending rates and to see how this is impacted by the nature of the donations process, the parameters describing university preferences and the endowment size. We have found that the usual substitution and wealth effects can be reversed, especially with respect to either very bad or good investment opportunities, as exemplified by smaller endowments in particular.
In contrast to the Campbell and Viceira (1999) model with zero donations, we observe the allocation to risky assets to be non-linear in expected risk premia. If donations depend positively on risky asset returns this can be explained by a substitution effect; donations implicitly increase risky asset holdings of the university, therefore reducing the optimal allocation to stocks. If investment opportunities are very good, though, endowments allocate substantially more to risky assets since the non-negativity constraint of donation inflows resembles an option to the university. The substitution effect is attenuated if expected risk premia are very low (negative) as expected donations converge to zero. Alternatively, if donations are positively associated with endowment returns the substitution effect vanishes. For low expected returns, endowments are more conservative in shorting the market as the expected utility gain from the combined effect of still possible donation inflows and portfolio returns is higher than in the original model without donations. Endogenous donation inflows give rise to an upward shift in the log consumption-to-wealth ratio. Moreover, optimal spending exhibits a wealth dependent minimum, which is especially true if donations depend on endowment returns. The optimal allocation to risky assets is mainly driven by variations in the parameter of risk aversion, while changes in the elasticity of intertemporal substitution have a strong effect on the log consumption-to-wealth ratio. Endowment size seems to correspond to a variation in the asset allocation profiles, as larger endowments have less of a convex asset allocation relation with respect to investment opportunities. Similarly, the spending rate of larger endowments is less volatile in terms of investment opportunities as compared to smaller endowments, especially when donations depend on the return to the optimal endowment portfolio.

We also look at US university endowment practices in recent years. We find support for our model incorporating time varying investment opportunities. Building in a model of donations inflows and its correlation with endowment returns and risky investment opportunities shows that universities behave on average as though a substitution effect between donations and investment returns takes place. Larger endowments do act in a more sensitive way to investment opportunities and donation flows as predicted by our model for the majority of investment states. Although larger endowments tend to spend a lower and less variable portion of their wealth, more donation intensity can permit higher spending rates.

While our paper contributes to the literature on endowment management by adapting the widely cited model of Campbell and Viceira (1999) to an endowment framework, there are various dimensions along which our work could be extended. Including several asset classes, most importantly alternative assets, would be a desirable extension relevant to endowment funds. Moreover, analyzing the impact of illiquid asset holdings on endowment spending might also be an interesting avenue for research. Modeling the interaction across universities as they compete with each other for reputational advantage would be a further useful extension, e.g., as in Goetzmann and Oster (2012). While we have considered (and verified empirically) a positive correlation, it is noteworthy that there may be negative correlations if the source of income is related to state support. That is, negative correlations may ensue if governments withdraw support to universities with superior endowment performance.

Our model contains numerous predictions about the investment management policies for university endowments and emphasizes that there is no “one size fits all” recommendation. However
this does not mean that there is complete freedom. The important aspect of university endowments is consistency between the donations process, asset allocation and spending rates. The model contained here has not only integrated these key parameters but has empirically tested observed policies to see how well they conform. Of course the ultimate objective is to see to what extent the results of this paper can be utilized for even better practices in the future by this important institutional investor class.

A Proof of Homogeneity of Value Function with Endogenous Donations

The university’s utility is

\[ U(C_t, \mathbb{E}_t U_{t+1}) = \left( (1-\delta) C_t^{(1-\rho)/\theta} + \delta \left( \mathbb{E}_t U_{t+1}^{1-\rho} \right)^{1/\theta} \right)^{\theta/(1-\rho)} \]  

where \( \theta = \frac{1-\rho}{1-\psi} \), \( \delta \) is the discount factor, \( \rho \) is the coefficient of relative risk aversion, \( C_t \) is consumption at time \( t \), \( \psi \) is the elasticity of intertemporal substitution and \( \alpha_t \) is the allocation to the risky asset at time \( t \). We prove this result in general for any nondecreasing measurable function of rates of return, \( f(\cdot) \), depending only on contemporaneous values of the state variable, \( x \). Hence, let the donation process is given by

\[ D_{t+1} = f(R_{t+1} - 1)(W_t - C_t) \]

and the agent’s budget constraint

\[ W_{t+1} = R_{t+1}(W_t - C_t) + D_{t+1} \]  

then the value function is proportional to wealth

\[ V_t = U_t(C_t, \mathbb{E}_t U_{t+1}) \].

To see this, divide equation (24) through by \( W_t \):

\[ \frac{U(C_t, \mathbb{E}_t U_{t+1})}{W_t} = \frac{\left( (1-\delta) C_t^{(1-\rho)/\theta} + \delta \left( \mathbb{E}_t U_{t+1}^{1-\rho} \right)^{1/\theta} \right)^{\theta/(1-\rho)}}{W_t^{1-\rho}} \]

\[ \frac{V_t}{W_t} = \frac{\left( (1-\delta) C_t^{(1-\rho)/\theta} + \delta \left( \mathbb{E}_t U_{t+1}^{1-\rho} \right)^{1/\theta} \right)^{\theta/(1-\rho)}}{W_t^{1-\rho}} \]

\[ \frac{V_t}{W_t} = \left( \frac{C_t}{W_t} \right)^{(1-\rho)/\theta} + \frac{\delta \left( \mathbb{E}_t U_{t+1}^{1-\rho} \right)^{1/\theta}}{W_t^{1-\rho}} \right)^{\theta/(1-\rho)} \].
Using the budget constraint (25) note that

\[ U_{t+1}^{(1-p)} = \left( \frac{U_{t+1}}{W_{t+1}} W_{t+1} \right)^{(1-p)} = \left( \frac{U_{t+1}}{W_{t+1}} \left( R_{t+1} + f(R_{t+1} - 1) \right) \right)^{(1-p)} \]

Hence,

\[ \delta \left( \mathbb{E}_t U_{t+1}^{(1-p)} \right)^{1/\theta} = \delta \left( \mathbb{E}_t \left[ \left( \frac{U_{t+1}}{W_{t+1}} \left( R_{t+1} + f(R_{t+1} - 1) \right) \right)^{1-p} \right] \right) W_{t}^{1-p} \]

Thus the following expression is obtained for utility per unit of wealth

\[ V_{t}/W_{t} = \left( 1 - \delta \left( \frac{C_{t}}{W_{t}} \right)^{(1-p)} \right) + \delta \left( 1 - \frac{C_{t}}{W_{t}} \right)^{(1-p)} \mathbb{E}_t \left[ \left( \frac{V_{t+1}}{W_{t+1}} \right)^{(1-p)} \left( R_{t+1} + f(R_{t+1} - 1) \right)^{(1-p)} \right]^{\frac{1}{\theta}} \]

Therefore it is sufficient to model the state space in one dimension, \( x_t \), with decision variables \( \alpha_t \) and \( \frac{C_{t}}{W_{t}} \) (spending rate).

B Verification of the Solution to Campbell-Viceira

We verify that our numerical procedure replicates the results of Campbell and Viceira (1999) for the case \( d = 0 \)\(^7\).

To verify our procedure we set \( d \) in equations (17) and (18) equal to 0, thus eliminating the effects of donations. Setting \( \rho \) equal to 4 and \( \psi \) equal to 0.1 we evaluate the value function to get the optimal values of \( \alpha_t, C_t/W_t \) and consequently the optimal log consumption-to-wealth ratio of our numerical solution. Conditional moments of the risky asset returns process are derived according to the computation in Appendix C. To obtain the corresponding levels of the control variables of the analytical solution, we use the set of optimal parameter values \( \{a_o, a_1, b_0, b_1, b_2\} \) in equations (21) and (22) as reported in Campbell and Viceira (1996)\(^8\).

\(^7\)Campbell, Cocco, Gomes, Maenhout, and Viceira (2001) also provide a numerical solution to the Campbell and Viceira (1999) framework. They use a different procedure for the optimization, namely the Newton-Raphson algorithm (a full-Newton method), whereas we use dynamic programming and BFGS.

\(^8\)Note, however, that the parameters reported in Campbell and Viceira (1996) are normalized so that the intercepts of the optimal policy functions correspond to the optimal \( \alpha \) and log consumption-to-wealth ratio when the expected discrete risk premium is zero. This involves a straightforward transformation of coefficients as explained there.
The results of our verification procedures are illustrated in Figure 12. This graph shows the Campbell-Viceira approximate solution (red) as compared to our numerical solution (blue). The green line is the best linear fit through our optimal solution. It is apparent that the approximate solution deviates from the slight degree of nonlinearity at the upper range of values for $x$. There is also a smaller deviation at the lower range of $x$-values. Near the midpoint of the $x$ range there is zero deviation. In general the slope of the asset allocation function is slightly greater in the approximate solution as compared to the fully optimal one.
Figure 13: Numerical vs. Analytical Solution

This chart plots the analytic approximation of the Campbell-Viceira consumption-wealth ratio, which appears in red color, against two solutions derived from our procedure. The blue curve is the numerical solution derived from the optimal solution procedure and the green line is a quadratic fit to this solution.

Similarly, Figure 13 illustrates the numerical solution against the approximate Campbell-Viceira solution for the spending rate. As before the Campbell-Viceira model is illustrated in red color, while our nonlinear solution is given by the blue curve - with the best quadratic fit in green. We find that the Campbell-Viceira solution slightly understates consumption in relation to wealth for small values of $x$ while overstating the precise solution for very high values.

We further validate our method by comparing means and standard deviations of the optimal control variables for both the numerical solution and the analytical solution. Furthermore we analyze average correlations of the optimal control variables for the two approaches over all simulated paths. Finally, we use the simulated paths of $x$ and the numerically optimized control variables to estimate the coefficients of equation (21) in a linear regression setup as well as fitting the coefficients in equation (22) quadratically.

The descriptive statistics provided in Table 11 show that the levels of the optimal $\alpha$ and consumption-to-wealth ratio are very close. Furthermore, the correlation between the two methods is close to one for both control variables. This suggests that our numerical procedure is indeed able to reproduce the results of the approximate analytical solution in Campbell and Viceira (1999) sufficiently well. Table 12
compares our estimates (D) of the optimal coefficients of the portfolio policy functions (equations 21 and 22) to those of Campbell and Viceira (1996) (CV). The coefficients are very similar in economic terms.

<table>
<thead>
<tr>
<th></th>
<th>25%</th>
<th>median</th>
<th>mean</th>
<th>75%</th>
<th>sd.</th>
<th>cor.</th>
<th>25%</th>
<th>median</th>
<th>mean</th>
<th>75%</th>
<th>sd.</th>
<th>cor.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CV</td>
<td>0.385</td>
<td>0.713</td>
<td>0.715</td>
<td>1.039</td>
<td>0.480</td>
<td>0.999</td>
<td>-3.018</td>
<td>-2.947</td>
<td>-2.918</td>
<td>-2.851</td>
<td>0.129</td>
<td>0.999</td>
</tr>
<tr>
<td>D</td>
<td>0.356</td>
<td>0.670</td>
<td>0.666</td>
<td>0.977</td>
<td>0.448</td>
<td>0.999</td>
<td>-2.977</td>
<td>-2.912</td>
<td>-2.885</td>
<td>-2.822</td>
<td>0.117</td>
<td>0.999</td>
</tr>
</tbody>
</table>

Table 11: Optimal Control Variables

This table provides descriptive statistics for the optimal control variables. The first row shows figures for the allocation to stocks and the log consumption-to-wealth ratio resulting from the analytically derived optimal policy functions (equations 21 and 22) in Campbell and Viceira (1999), abbreviated by CV, while the second row shows the corresponding figures using our numerical optimization procedure (D). We report the 25% and 75% quantile, mean, median, standard deviation (sd.) and correlation (cor.).

<table>
<thead>
<tr>
<th></th>
<th>25%</th>
<th>median</th>
<th>mean</th>
<th>75%</th>
<th>sd.</th>
<th>cor.</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₀</td>
<td>0.302</td>
<td>9.710</td>
<td>-3.032</td>
<td>1.504</td>
<td>11.58</td>
<td></td>
</tr>
<tr>
<td>a₁</td>
<td>0.277</td>
<td>8.783</td>
<td>-2.984</td>
<td>1.359</td>
<td>9.944</td>
<td></td>
</tr>
</tbody>
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Table 12: Coefficients of the Policy Function

This table provides the optimal coefficients for the portfolio policy functions. a₀ and a₁ are the coefficients of the linear relation between the allocation to the risky asset and expected risk premia, whereas b₀, b₁ and b₂ are the coefficients of the quadratic relation between the log consumption-to-wealth ratio and expected risk premia. The first row provides the coefficients taken from Campbell and Viceira (1996), abbreviated by CV, while the second row shows the coefficients resulting from regressions of the numerically optimized control variables on expected risk premia (D).

C Derivation of the Conditional Returns Process

Variables known at time t are: xₜ and rₜ, whereas at time t + 1 we know: xₜ₊₁ and rₜ₊₁. To fit the appropriate return process into the dynamic programming approach we need to calculate the conditional distribution

\[ (r_{t+1}, x_{t+1}) | (r_t, x_t) \]

From the assumptions set up in section 2 we know that

\[ x_t \sim N(\mu, \frac{\sigma^2}{1-\phi^2}) \]

\[ (r_{t+1}) | (x_t) \sim N(x_t + r_f, \sigma^2_u) \]

\[ \mathbb{E}(r_{t+1}) = \mathbb{E}(\mathbb{E}(r_{t+1} | x_t)) = \mathbb{E}(x_t + r_f) = \mu + r_f \]

\[ \sigma^2(r_{t+1}) = \sigma^2(x_t) + \sigma^2(u_{t+1}) = \frac{\sigma^2_\eta}{(1-\phi)^2} + \sigma^2_u \]

\[ \text{cov}(r_{t+1}, x_t) = \text{cov}(x_t + r_f + u_{t+1}, x_t) = \frac{\sigma^2_\eta}{(1-\phi)^2} \]
using \( x_t = (1 - \phi)\mu + \phi \cdot x_{t+1} + \eta_t \)

Then the multivariate distribution of \((r_{t+1}, x_t)\) is

\[
\begin{pmatrix}
  r_{t+1} \\
  x_t
\end{pmatrix}
\sim N
\begin{pmatrix}
  \mu + r_f \\
  \mu + r_f
\end{pmatrix},

\begin{pmatrix}
  \frac{\sigma^2}{(1 - \phi^2)} + \sigma_u^2 & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} \\
  \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u}
\end{pmatrix}
\]

The next step is to calculate the multivariate distribution of \((r_{t+1}, x_{t+1}, r_t, x_t)\) which is

\[
\begin{pmatrix}
  r_{t+1} \\
  x_{t+1} \\
  r_t \\
  x_t
\end{pmatrix}
\sim N
\begin{pmatrix}
  \mu + r_f \\
  \mu + r_f \\
  \mu + r_f \\
  \mu
\end{pmatrix},

\begin{pmatrix}
  \frac{\sigma^2}{(1 - \phi^2)} + \sigma_u^2 & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} \\
  \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} \\
  \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} \\
  \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \phi \cdot \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} & \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u}
\end{pmatrix}
\]

This follows from the following calculations:

\[
\mathbb{E}(r_{t+1}) = \mathbb{E}(r_t) = \mu + r_f \quad \text{and} \quad \sigma^2(r_{t+1}) = \sigma^2(r_t) = \frac{\sigma_{\eta}^2}{1 - \phi^2} + \sigma_u^2
\]

\[
\mathbb{E}(x_{t+1}) = \mathbb{E}(x_t) = \mu \quad \text{and} \quad \sigma^2(x_{t+1}) = \sigma^2(x_t) = \frac{\sigma_{\eta}^2}{1 - \phi^2}
\]

\[
cov(r_{t+1}, r_t) = cov(x_t + r_f + u_{t+1}, x_{t-1} + r_f + u_t) =
\]

\[
= cov((1 - \phi)\mu + \phi x_{t-1} + \eta_t + r_f + u_{t+1}, x_{t-1} + r_f + u_t) =
\]

\[
= \phi \sigma(x_{t-1}, x_{t-1}) + \sigma(\eta_t, u_t) = \phi \frac{\sigma_{\eta}^2}{1 - \phi^2} + \sigma_{\eta u}
\]

\[
cov(r_{t+1}, x_{t+1}) = cov(x_t + r_f + u_{t+1}, (1 - \phi)\mu + \phi x_t + \eta_{t+1}) =
\]

\[
= \phi \frac{\sigma_{\eta}^2}{1 - \phi^2} + \sigma_{\eta u}
\]

\[
cov(x_{t+1}, x_t) = cov((1 - \phi)\mu + \phi x_t + \eta_{t+1}, x_t) = \phi \frac{\sigma_{\eta}^2}{1 - \phi^2}
\]

44
\[ \text{cov}(r_t, x_t) = \text{cov}(x_{t-1} + r_f + u_t, (1 - \phi)\mu + \phi x_{t-1} + \eta_t) = \]
\[ = \text{cov}(r_{t+1}, x_{t+1}) = \phi \frac{\sigma^2}{(1 - \phi^2)} + \sigma_{\eta u} \]

\[ \text{cov}(r_t, x_{t+1}) = \text{cov}(x_{t-1} + r_f + u_t, (1 - \phi)\mu + \phi x_t + \eta_{t+1}) = \]
\[ = \text{cov}(x_{t-1} + r_f + u_t, (1 - \phi)\mu + \phi((1 - \phi)\mu + \phi x_{t-1} + \eta_t + \eta_{t+1}) = \]
\[ = \text{cov}(x_{t-1} + r_f + u_t, (1 - \phi)\mu + \phi(1 - \phi)\mu + \phi^2 x_{t-1} + \phi \eta_t + \eta_{t+1}) = \]
\[ = \phi^2 \text{cov}(x_{t-1}, x_{t-1}) + \phi \text{cov}(u_t, \eta_t) = \]
\[ = \phi^2 \frac{\sigma^2}{(1 - \phi^2)} + \phi \sigma_{\eta u} \]

To calculate the conditional distribution \((r_{t+1}, x_{t+1})| (r_t, x_t)\) note that for multivariate random variables the following rules apply:

If \(x\) is a \(N\)-dimensional random vector multivariate normally distributed \((x \sim \mathcal{N}(\mu, \Sigma))\) and partitioned as follows

\[
\mu = \begin{pmatrix} 
\mu_1 \\
\mu_2
\end{pmatrix}
\]

with sizes

\[
\begin{pmatrix} 
q \times 1 \\
(N - q) \times 1
\end{pmatrix}
\]

and

\[
\Sigma = \begin{pmatrix} 
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}
\]

with sizes

\[
\begin{pmatrix} 
q \times q & q \times (N - 1) \\
(N - q) \times q & (N - q) \times (N - q)
\end{pmatrix}
\]

then the mean and variance-covariance matrix of the conditional distribution of \(x_1|x_2 = a\) is given by

\[ \tilde{\mu} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1}(a - \mu_2) \]
and
\[ \Sigma = \Sigma_{11} + \Sigma_{12} \Sigma_{22} \Sigma_{21} \]

For the sake of notional brevity we use the following abbreviations

\[ A = \frac{\sigma_\eta^2}{1 - \phi^2}, B = \sigma_u^2, C = \sigma_{\eta u} \]

Hence,
\[
\begin{pmatrix}
  r_{t+1} \\
  x_{t+1} \\
  r_t \\
  x_t
\end{pmatrix}
\sim
N
\left( \begin{pmatrix}
  \mu + r_f \\
  \mu + r_f \\
  \mu \\
  \mu
\end{pmatrix},
\begin{pmatrix}
  A + B & \phi A + C & \phi A + C & A \\
  \phi A + C & A & \phi^2 A + \phi C & \phi A \\
  \phi A + C & \phi^2 A + \phi C & A + B & \phi A + C \\
  A & \phi A & \phi A + C & A
\end{pmatrix} \right)\
\]

\[ \hat{\mu} = \begin{pmatrix}
  \mu + r_f \\
  \mu
\end{pmatrix} + \Sigma_{12} \Sigma_{22}^{-1} \begin{pmatrix}
  r_t - (\mu + r_f) \\
  x_t - \mu
\end{pmatrix} \]

\[ \Sigma_{12} = \begin{pmatrix}
  \phi A + C & A \\
  \phi^2 A + \phi C & \phi A
\end{pmatrix} \quad \Sigma_{22} = \begin{pmatrix}
  A + B & \phi A + C \\
  \phi A + C & A
\end{pmatrix} \]

\[ \Sigma_{22}^{-1} = \frac{1}{(A + B)A - (\phi A + C)^2} \begin{pmatrix}
  A & -(\phi A + C) \\
  -(\phi A + C) & A + B
\end{pmatrix} \]

Multiplying \( \Sigma_{12} \) with \( \Sigma_{22}^{-1} \) and setting \( D = (A + B)A - (\phi A + C)^2 \)

\[ \Sigma_{12} \Sigma_{22}^{-1} = \frac{1}{D} \begin{pmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  0 & \phi
\end{pmatrix} \]

with

\[ M_{11} = (\phi A + C)A - (\phi A + C)A = 0 \]

\[ M_{12} = (A + B)A - (\phi A + C)^2 = D \]

\[ M_{21} = A(\phi^2 A + \phi C) - \phi A(\phi A + C) = \phi^2 A^2 + \phi AC - \phi^2 A^2 - \phi AC = 0 \]

\[ M_{22} = -(\phi A + C)(\phi^2 A + \phi C) + \phi A(A + B) = -\phi^2 A^2 - \phi^2 AC - \phi^2 AC - \phi C^2 + \phi A^2 + \phi BA = \]

\[ = \phi((\phi^2 A^2 - 2\phi AC - C^2) + A(A + B)) = \phi((\phi A + C)^2 + A(A + B)) = \phi D \]
\[
\hat{\mu} = \begin{pmatrix} \mu + r_f \\ \mu \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & \phi \end{pmatrix} \begin{pmatrix} r_t - (\mu + r_f) \\ x_t - \mu \end{pmatrix}
\]

\[
\hat{\mu} = \begin{pmatrix} \mu + r_f + x_t - \mu \\ \mu + \phi(x_t - \mu) \end{pmatrix} = \begin{pmatrix} r_f + x_t \\ \mu + \phi(x_t - \mu) \end{pmatrix}
\]

And for the Variance-Covariance matrix

\[
\hat{\Sigma} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}
\]

\[
\Sigma_{11} = \begin{pmatrix} A + B & \phi A + C \\ \phi A + C & A \end{pmatrix}, \quad \Sigma_{21} = \begin{pmatrix} \phi A + C & \phi^2 A + \phi C \\ A + \phi A \\ A + \phi A \end{pmatrix}
\]

\[
\hat{\Sigma} = \begin{pmatrix} A + B & \phi A + C \\ \phi A + C & A \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & \phi \end{pmatrix} \begin{pmatrix} \phi A + C & \phi^2 A + \phi C \\ A & \phi A \end{pmatrix} = \begin{pmatrix} A + B - A & \phi A + C - \phi A \\ \phi A + C - \phi A & A - \phi^2 A \end{pmatrix} = \begin{pmatrix} B & C \\ C & A - \phi^2 A \end{pmatrix}
\]

Note that

\[
B = \sigma_u^2
\]

\[
C = \sigma_{\eta u}
\]

\[
A - \phi^2 A = \frac{\sigma_\eta^2}{1 - \phi^2} - \phi^2 \frac{\sigma_{\eta u}^2}{1 - \phi^2} = \frac{\sigma_\eta^2}{1 - \phi^2} (1 - \phi^2) = \sigma_{\eta u}^2
\]

And so the conditional distribution of \((r_{t+1}, x_{t+1})|(r_t, x_t)\) is given by

\[
(r_{t+1}, x_{t+1})|(r_t, x_t) \sim N \left( \begin{pmatrix} r_f + x_t \\ (1 - \phi) \mu + \phi x_t \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{\eta u} \\ \sigma_{\eta u} & \sigma_{\eta u}^2 \end{pmatrix} \right)
\]

which is independent of \(r_t\), hence \(x_t\) is sufficient at \(t\) to know about the conditional distribution of \((r_{t+1}, x_{t+1})\), thus

\[
(r_{t+1}, x_{t+1})|(r_t, x_t) = (r_{t+1}, x_{t+1})|(x_t)
\]
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