Why Fundamental Indexation Might—or Might Not—Work

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Some proponents of fundamental indexation claim that the strategy is based on a new theory in which market prices of stocks deviate from fair values. A key assumption in this approach is that fundamental weights are unbiased estimators of fair value weights that are statistically independent of market values. This article demonstrates that, except in trivial cases, this assumption is internally inconsistent because the sources of the "errors" are also determinants of market values. The article shows under what conditions fundamental weights are better—or worse—estimators of fair value weights than are market value weights, thereby demonstrating that the new theory is merely a conjecture. A formula is developed for the value bias inherent in fundamental weighting, and two approaches to combining fundamental and market values are discussed.

Some proponents of fundamental indexation claim that their strategy is based on a revolutionary new paradigm in which market prices of stocks deviate from their fair values.[1] The new paradigm is called the "noisy market hypothesis."[2] And it is said to be in opposition to the efficient market hypothesis, in the sense that its proponents believe that a fundamentally weighted index is a better representation of fair market values than is a capitalization-weighted index.

Skeptics are quick to argue that fundamental indexation is nothing more than value investing in a different guise (e.g., Asness 2006, 2007a, 2007b). Proponents claim that their approach is different from value investing because it takes advantage of the noise in stock prices rather than taking advantage of value premiums. As Asness pointed out in the works cited, however, previous researchers, such as Lakonishok, Shleifer, and Vishny (1994), had already shown that noise in stock prices can serve as the rationale for value investing.[3]

Stated simply, the problem to be addressed is this: If the measure of fundamental value is, say, earnings, proponents of fundamental indexing assume that a good approximation is that all companies should sell at the same price-to-earnings ratio (P/E). Proponents of market-cap-weighted

indexing, in contrast, assume that the market is correct in its setting of different—sometimes widely different—P/Es.

Fundamental indexers are clearly wrong in their assertion on first principles, because risk and growth, the determinants of a fair P/E, do vary among companies. Yet, market-cap proponents also may be empirically wrong if the market *errs* in setting different P/Es for different companies—as it does.

Perold (2007) examined the logic of the noisy-market hypothesis and found it flawed. He pointed out that a key assumption of the model is that investors do not know the fair values of stocks; they know only the market prices and the probability *distributions* from which the pricing errors are drawn. However, when the proponents of fundamental indexation formulate the theoretical expected returns of market-cap-weighted portfolios in their publications, they include the fair values in the information set. In other words, they implicitly, and perhaps unknowingly, assume that the investor knows the fair value of the stock. Not surprisingly, they find a "drag" in the expected return of the market-cap-weighted portfolio because stocks known to be overvalued are overweighted and stocks known to be undervalued are underweighted. When Perold formulated expected returns without knowledge of the fair prices, however, the drag disappeared.

In this article, I explore the logic of fundamental indexation from another angle—the concept of fair value multiples. The fair value multiple is the

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(unobservable) number M^* that equates the (unobservable) fair value of a stock, V^*, with some observed measure of the stock's fundamental value, F, as follows: $V^* = FM^*$ or $M^* = V^*/F$.

Proponents of fundamental indexation assert that fundamental weights can be unbiased estimators of the unobservable fair value weights with "errors" that are statistically independent of market values. I refer to this assertion as the "independence assumption" and demonstrate that it is internally inconsistent. The so-called errors are actually restatements of the fair value multiples of the stocks in question. For example, if the fundamental weights are based on earnings, the errors in the fundamental weights are restatements of the fair P/Es of the stocks.

A stock's fair value multiple, by definition, reflects investors' assessments of the stock's risk and future growth prospects. Ideally, such factors should be fully taken into account in portfolio construction. Because they are unobservable, however, they must be either taken into account through proxies or ignored. Market-cap weighting takes risk and expected growth into account by using the market values of stocks as proxies for their unobservable fair values. If market prices contain noise, market-cap weights contain errors. Fundamental weighting schemes introduce weighting errors of a different type by ignoring risk and expected growth. The superior weighting scheme is the one with the less egregious type of error.

I demonstrate that, except in trivial cases, fundamental weights cannot be unbiased estimators of fair value weights with errors that are statistically independent of market values because the sources of those errors, risk and expected growth, are also determinants of market values. I do so formally by showing that the errors are restatements of fair value multiples and that fair value multiples are correlated with market values unless all stocks have the same fair value multiple. Then, I show the conditions under which fundamental weights are better than market value weights as estimators of fair value weights and the conditions under which they are worse. I develop a formula for the value bias inherent in fundamental weighting. And finally, I consider how fundamental weights and market value weights could be combined to form a better estimator of fair value weights than either one separately.

The Independence Assumption

One of the leading proponents of fundamental indexation, Hsu (2006, p. 49), stated the independence assumption as follows:

Suppose we use some other measure of firm size instead of market capitalization to create portfolio weights. For simplicity, let us suppose we do an admirable job at creating the weights so that

$$\widehat{w}_{i,t} = w^*_{i,t}\left(1 + v_{i,t}\right)$$

Where $v_{i,t}$ is a mean zero white noise uncorrelated with other random variables. [$w^*_{i,t}$ is the weight of stock i at time t based on fair values and $\widehat{w}_{i,t}$ is the fundamental weight.] This is to say that the selected portfolio weights may deviate significantly and across the board from the "true-value-weight," but these mistakes in assigning weights are not related to other variables, such as *market prices or firm capitalization.* [Emphasis added.]

Now, let us see if fundamental weights can satisfy these criteria. We use the following variables:

$F_{i,t}$ = fundamental measure of size used to construct $\widehat{w}_{i,t}$

$V^*_{i,t}$ = true fair value of stock i at time t

N = number of stocks in the portfolios being considered

By definition,

$$\widehat{w}_{i,t} = \frac{F_{i,t}}{\sum_{j=1}^{N} F_{j,t}} \tag{1}$$

and

$$w^*_{i,t} = \frac{V^*_{i,t}}{\sum_{j=1}^{N} V^*_{j,t}}. \tag{2}$$

Hence,

$$1 + v_{i,t} = \frac{F_{i,t}}{V^*_{i,t}} \frac{\sum_{j=1}^{N} V^*_{j,t}}{\sum_{j=1}^{N} F_{j,t}}. \tag{3}$$

Note that the term $F_{i,t}/V^*_{i,t}$ in Equation 3 is the reciprocal of the fair value multiple of stock i at time t. So, the fair value multiple appears on the right-hand side of Equation 3 in spite of Hsu's (2006) assertion that the left-hand side is "white noise uncorrelated with other random variables . . . such as market prices or company capitalization." Hsu's assertion cannot hold unless fair value multiples are constant across stocks or market valuations are completely unrelated to fair values—two unimaginable scenarios. Fair value multiples should have *some* correlation with market values. After all, a stock's fair market multiple is related to its risk and prospective growth, which should be to some extent reflected in its market price.

By way of analogy, imagine that we have a collection of numerous gems of various types, qualities, and weights. We find out the market value of

each gem, write it on a bag, place the gem in the bag, and seal the bag. Once we have sealed the bags, we cannot tell which gem is in which bag. We do have a scale, however, so we weigh each bag and write the weight on the bag as well as the market value. As with stocks, for these gems, we know the market value and have a fundamental measure of size (weight) but we do not know the fair value or fair value multiple (fair price per ounce) for each variety of gem, each of which is likely to be different. Unless all the gems are of the same type and quality, however, or market values are completely unrelated to fair values, there is a relationship between the market values written on the bags and the fair prices per ounce of the gems inside the bags. So, we do not want to rely on weight alone to assess the value of each bag's content.

Proponents of fundamental indexing are asking us to rely solely on weight, and they assure us that any errors in valuation that this practice causes are unrelated to what types of gems are in the bags.

Why might such an idea seem plausible? Many investors believe that the ways in which the market values companies are not perfectly related to the companies' fair values. And these investors want to exploit the difference. They also want to achieve approximate macroconsistency with the market as a whole, however, and keep tracking error versus a cap-weighted benchmark to an acceptable level. Hence, they need to weigh their portfolios by some measure of size while trying to avoid the errors in market valuation. But doing so with fundamental weights ignores whatever additional information about fair values is contained in market prices. The independence assumption made by the proponents of fundamental indexation asserts that market prices are useless to such investors and can be safely ignored.

To demonstrate the fallacy of the independence assumption formally, let us decompose a stock's fair value, V^*, into two independent components: a fundamental measure of size, F, and the corresponding fair value multiple, M^*, as noted earlier. Dropping the subscript i from the earlier notation, we have the fair value multiple of a stock related to Hsu's "mistake" term as follows:

$$M^* = \frac{1}{1+\nu} \frac{\sum_{j=1}^{N} V_{j,t}^*}{\sum_{j=1}^{N} F_{j,t}}. \tag{4}$$

The fair value is simply the product of the fundamental value and the fair value multiple:

$$V^* = FM^*. \tag{5}$$

Like Hsu, we assume that a stock's market value differs from its fair value by an independent multiplicative error term, which Hsu denoted $1 + \varepsilon_{i,t}$ and we denote U. Hence,

$$V = V^*U. \tag{6}$$

Thus,

$$V = FM^*U. \tag{7}$$

My critique of Hsu is that he assumed that M^* is independent of FM^*U; that is, he assumed that $\text{cov}(M^*, FM^*U)$ is zero. Because F, M^*, and U are all independent of one another, however, and because $E(U)$ is equal to 1, then

$$\text{cov}\left(M^*, FM^*U\right) = E(F)\text{var}\left(M^*\right). \tag{8}$$

Because fundamental variables take only positive values, this covariance can be zero *only* if M^* is a constant. Otherwise, it is positive and the independence assumption is violated.

To sum up, this mathematical exercise demonstrates that Hsu's arguments start with math and logic that is internally inconsistent. Therefore, the conclusions that he draws do not necessarily follow. In particular, his conclusion that a fundamentally weighted index has a higher expected return than a market-weighted index lacks the theoretical foundation that he claims.

This is not to say that a fundamentally weighted index will not outperform a market-weighted index. Fundamental indexing should outperform in periods when well-crafted value strategies outperform market-cap benchmarks—and such periods seem to be the rule rather than the exception. The theoretical argument that its proponents put forth, however, is specious.

Why Fundamental Indexation *Might* Work

Having dismissed the notion that fundamental indexation *must* work, I will now show why it *might* work (and why it might not). Like Treynor (2005), my analysis here focuses on the relationships among fair prices, pricing errors, and market prices. Unlike Treynor, I assume that investors do not know the pricing errors of individual stocks.[4] Rather, they know only the probability distributions of the errors. Also, rather than trying to develop an explicit formula for how much value a non-market-weighting scheme might create for an investor above that of a market-cap-weighted portfolio of the same stocks, I develop a boundary condition that needs to be satisfied in order for a non-market-weighting scheme to add positive value.

If a fundamentally weighted portfolio is to outperform a market-cap-weighted portfolio of the same stocks, the fundamental variables used to construct the weights should contain more information about the fair values of the stocks than the market values of the stocks contain. My approach to assessing alternative weighting schemes is to compare (1) the correlation between the fundamental values and the fair values of the stocks in a portfolio with (2) the correlation between the market values and fair values. I then derive a necessary condition for the former correlation to exceed the latter correlation.[5] If the correlation between the fundamental values and the fair values exceeds the correlation between the market values and fair values, then fundamental indexation is the *a priori* superior approach.[6] If the reverse is true, then market-cap weighting is superior.

I use the natural logarithms of the variables in the analysis to simplify the formulas for the correlations. Because the fair value of a stock is the product of a fundamental measure of size, F, and the fair value multiple with respect to that fundamental measure, M^*, using lower-case letters to denote the natural logarithm of each of these variables, we have

$$v^* = f + m^*. \tag{9}$$

Similarly, because market value V is the product of fair value V^* and the multiplicative error term, U, we have

$$\begin{aligned} v &= v^* + u \\ &= f + m^* + u. \end{aligned} \tag{10}$$

The three variables f, m^*, and u are mutually independent random variables. Their variances are denoted $\sigma^2(f)$, $\sigma^2(m^*)$, and $\sigma^2(u)$.

I assess the alternative weighting schemes by the correlations of the logarithms of their weights with the logarithms of fair value, v^*. For the fundamental-weighting scheme, this measure is the correlation of f and v^*. From the definition of correlation, the identity stated in Equation 9, and the independence of f and m^*, it follows that the correlation of f and v^* is

$$\mathrm{corr}(f, v^*) = \frac{\sigma(f)}{\sqrt{\sigma^2(f) + \sigma^2(m^*)}}. \tag{11}$$

Note that this correlation is 1 if there is no variation in fair value multiples.

For market-cap weighting, the assessment is based on the correlation between v and v^*. From the definition of correlation, the identities stated in

Equations 9 and 10, and the mutual independence of f, m^*, and u, it follows that the correlation of v and v^* is

$$\mathrm{corr}(v, v^*) = \sqrt{\frac{\sigma^2(f) + \sigma^2(m^*)}{\sigma^2(f) + \sigma^2(m^*) + \sigma^2(u)}}. \tag{12}$$

Note that this correlation is 1 if market values are error free.

So, we will judge fundamental weighting to be superior to market-cap weighting if

$$\mathrm{corr}(f, v^*) > \mathrm{corr}(v, v^*). \tag{13}$$

From Inequality 13, we can calculate the lowest value of the variability of pricing errors, $\sigma(u)$, that is consistent with fundamental weighting being superior to market-cap weighting. To do this, we use the formulas for $\mathrm{corr}(f, v^*)$ and $\mathrm{corr}(v, v^*)$ given in Equations 11 and 12 so that Inequality 13 becomes

$$\frac{\sigma(f)}{\sqrt{\sigma^2(f) + \sigma^2(m^*)}} > \sqrt{\frac{\sigma^2(f) + \sigma^2(m^*)}{\sigma^2(f) + \sigma^2(m^*) + \sigma^2(u)}}. \tag{14}$$

Bringing $\sigma(u)$ to the left-hand side simplifies the inequality to

$$\sigma(u) > \sigma(m^*)\sqrt{1 + \frac{\sigma^2(m^*)}{\sigma^2(f)}}. \tag{15}$$

Inequality 15 is thus a boundary condition for fundamental weighting to be superior to market-cap weighting: The valuation errors that the market makes must be more variable than fair value multiples.

Conversely, if the reverse of Inequality 15 is true, so that

$$\sigma(u) < \sigma(m^*)\sqrt{1 + \frac{\sigma^2(m^*)}{\sigma^2(f)}}, \tag{16}$$

then market-cap weighting is superior to fundamental weighting. That is, if the market's value errors across stocks are less variable than fair value multiples, weighting by market value is better than weighting by fundamental value.

Because there is no theoretical basis for deciding whether Inequality 15 or Inequality 16 holds, there is no theoretical foundation for fundamental indexation. Furthermore, because none of the variables in these inequalities are observable, one cannot devise empirical tests, apart from historical performance, to see whether fundamental weighting or market-cap weighting is a better strategy. In other words, fundamental indexation is based on a conjecture about unobservable variables. There is no theory to support it.

The Value Bias

As I mentioned in the introduction, some critics of fundamental indexation point out that fundamental weighting is simply an alternative way to introduce a value bias into a portfolio. Because value-biased portfolios historically have outperformed unbiased portfolios, it is no surprise that a fundamentally weighted index outperforms a market-cap index of the same stocks in a long-term backtest.[7] In this section, I develop a formula for assessing the extent of the value bias in a portfolio weighted by a single fundamental size measure. (The analysis can be extended to weighting schemes that use multiple fundamental measures.)

Again, N is the number of stocks in the portfolio, V_i is the market value of stock i, and F_i is the fundamental size measure of stock i. The yield (dividend yield, earnings yield, etc.) on stock i is

$$y_i = \frac{F_i}{V_i}. \tag{17}$$

Let w_i be the weight of stock i in the market-value-weighted portfolio and x_i be the weight of stock i in the fundamentally weighted portfolio. Hence,

$$w_i = \frac{V_i}{\sum_{j=1}^{N} V_j} \tag{18}$$

and

$$x_i = \frac{F_i}{\sum_{j=1}^{N} F_j}. \tag{19}$$

The yield on the market-value-weighted index is

$$\begin{aligned} y_V &= \sum_{j=1}^{N} w_j y_j \\ &= \frac{\sum_{j=1}^{N} F_j}{\sum_{j=1}^{N} V_j}. \end{aligned} \tag{20}$$

From Equations 18, 19, and 20, we have

$$x_i = \frac{y_i}{y_V} w_i. \tag{21}$$

In other words, as Asness (2007b) pointed out, the fundamental weight is simply the market-cap weight multiplied by the ratio of the yield on the security to the yield on the market-cap-weighted portfolio. Hence, the fundamentally weighted index underweights stocks that have lower yields than the market-cap-weighted portfolio and overweights stocks that have higher yields than the market-cap-weighted portfolio. The underweighting and overweighting are in proportion to each stock's yield.

This fact, then, is the source of the value bias in fundamental indexation. This analysis, however, has examined it only at the security level, but the analysis can be extended to the portfolio level.

The yield on the fundamentally weighted *index* is

$$y_F = \sum_{j=1}^{N} x_j y_j. \tag{22}$$

From Equations 21 and 22, we have

$$y_F = \frac{\sum_{j=1}^{N} w_j y_j^2}{y_V}. \tag{23}$$

The market-value-weighted variance of yields across the stocks is

$$\sigma_y^2 = \sum_{j=1}^{N} w_j \left(y_j - y_V \right)^2 = \sum_{j=1}^{N} w_j y_j^2 - y_V^2. \tag{24}$$

From Equations 22 and 23, we have

$$y_F - y_V = \frac{\sigma_V^2}{y_V}. \tag{25}$$

Hence, the yield on a fundamentally weighted index exceeds the yield on a market-value-weighted index of the same stocks by the ratio of the variance of yield across the stocks in the index to the yield on the market-weighted index.

Note that $y_F \geq y_V$. This is not a boundary condition; it is true by construction (because σ_V^2 must be greater than or equal to zero). Thus, if a value bias is defined as a portfolio yield higher than that of the market-cap-weighted index, the conclusion that fundamental indexation contains a value bias does not depend on circumstances (that is, on the observed values of variables). It is always true.

Toward a Better Weighting Method

Because fundamental data and market capitalizations *both* contain information about fair market value, the best thing to do would be to *combine* them to form estimates of fair market value. In this section, I briefly discuss two approaches to combining the methods. One is based on a theoretical optimization. The other is a pragmatic approach that tries to combine the best features of both approaches while avoiding the disadvantages of each.

Optimization Approach. The first approach builds off the previous analysis of the components of fair value, fundamental value, and market value. Recall that the criterion for ranking weighting-scheme variables is the correlation of the logarithm of the variables with the logarithm of fair value, v^*.

Because the logarithms of fundamental value f and market value v are both correlated with v^*, there could be linear combinations of the two that have a higher correlation than either one separately. In Appendix A, I show that the linear combination of f and v that has the highest correlation with v^* is

$$\hat{v} = \theta f + (1-\theta)v, \qquad (26)$$

where

$$\theta = \frac{\sigma^2(u)}{\sigma^2(m^*) + \sigma^2(u)}. \qquad (27)$$

The formula in Equation 27 echoes my previous observation that if there is no variation in fair value multiples across stocks [so that $\sigma(m^*) = 0$], fundamental weighting is optimal ($\theta = 1$). It also implies that if market values are error free [$\sigma(u) = 0$], market-cap weights are optimal ($\theta = 0$). If neither condition holds, the optimal weights fall somewhere between these two extremes, with the blended weights taking the form

$$\hat{w}_i = \frac{F_i^\theta V_i^{1-\theta}}{\sum_{j=1}^{N} F_j^\theta V_j^{1-\theta}}. \qquad (28)$$

Of course, the true value of θ is unobservable, so selecting it in practice would largely be a matter of judgment.

"Collared" Approach. Arya and Kaplan (2006, 2007) presented another approach to combining market value and fundamental variables, an approach that is not based on optimization. They developed a weighting methodology intended to combine the best practical features of the market value and fundamental approaches while avoiding the disadvantages of each.

The advantage of market-cap weighting that Arya and Kaplan sought to preserve is low turnover. A great advantage of market value weighting over any non-cap-weighted approach is that market value weighting does not need to be continuously rebalanced, so it keeps transaction costs to a minimum. In contrast, any non-cap-weighted scheme requires regular rebalancing and thus generates transaction costs that must be overcome by superior performance.

Weighting solely on market value, however, can produce portfolios with volatile levels of diversification. Cap-weighted portfolios are never rebalanced, so a stock that keeps rising has an increasingly large portfolio weight. Most investors would never allow this decreasing diversification at the asset-class level; rebalancing to a set of market-invariant asset-class weights keeps a port-

folio diversified. In a similar fashion, by rebalancing a portfolio of *stocks* to a set of market-invariant weights, fundamental indexation promotes diversification (to be precise, it reduces the volatility of the level of diversification).

To retain low turnover but control the level of diversification, Arya and Kaplan used fundamental weights to set *boundaries* on portfolio weights rather than to be the portfolio weights themselves. These boundaries (or collars) are fixed multiples of the fundamental weights, such as a lower bound of half the fundamental weight and an upper bound of twice the fundamental weight. Stocks with market value weights that fall within the boundaries are held at the market value weights, whereas stocks with market value weights that fall outside the boundaries are held at the boundary weights. The idea is to hold *most* of the portfolio at market value weights *most of the time* while avoiding concentration in a few large stocks during market run-ups, such as occurred during the late 1990s. The effect would be similar to the way in which strategic asset allocation models kept overall portfolios balanced during that period.

Conclusion

Proponents of fundamental indexation claim that the noisy-market hypothesis implies that fundamental weighting must be superior to market-cap weighting. Perold (2007) and this article demonstrate, in different ways, that these proponents' reasoning is flawed. Perold showed that they err in their formulation of the expected return of a market-cap-weighted portfolio. I have shown that one of their key assumptions, the independence assumption, must be false except in the most trivial cases. The proponents of fundamental indexation, rather than having a clear theory on which to base their claims, have only a conjecture that market valuation errors are more variable than fair value multiples. They may have been correct over long historical periods, as the successful backtests of their strategies seem to demonstrate, but they should be much more modest in their claims. In particular, they could argue that it is better to introduce a value bias into a portfolio by using their weighting scheme than by excluding low-yield stocks. But they have not, to date, pursued this more humble line of reasoning. They may have a successful investment strategy, but they have not produced a revolution in investment theory.

Finally, I showed that if market capitalization and fundamental variables both contain information about fair market values, they can be combined

to form portfolio weights that reflect the information contained in each. I discussed two ways to do so, but other approaches are possible and I encourage this line of research.

This article qualifies for 1 CE credit.

Appendix A. Derivation of Optimal Combination of Fundamental and Market Values

Recall the following definitions:

v = natural logarithm of market value

v^* = natural logarithm of fair value

f = natural logarithm of fundamental value

m^* = natural logarithm of fair value multiple

u = natural logarithm of the multiplicative pricing error in market value

$\hat{v}$ = linear combination of v and f (chosen to have maximum correlation with v^*)

$\sigma(x)$ = standard deviation of x (x being one of the preceding variables)

θ = weight on f in the construction of $\hat{v}$

The aim is to select θ so as to maximize the correlation between $\hat{v}$ and v^*. To simplify the algebra, we define the objective function of this problem, $Q(\cdot)$, as the natural logarithm of this correlation. (We can do this because the correlation between $\hat{v}$ and v^* at and around the optimal choice for θ is positive.) We have

$$
\begin{aligned}
Q(\theta) &= \ln\left[\frac{\text{cov}\left(\hat{v}, v^*\right)}{\sigma\left(\hat{v}\right)\sigma\left(v^*\right)}\right] \\
&= \ln\left[\text{cov}\left(\hat{v}, v^*\right)\right] - \frac{1}{2}\ln\left[\sigma^2\left(\hat{v}\right)\right] \\
&\quad - \ln\left[\sigma\left(v^*\right)\right].
\end{aligned}
\tag{A1}
$$

Recall that, by definition, the following three identities hold

$$
v^* = f + m^*
\tag{A2}
$$

$$
v = f + m^* + u
\tag{A3}
$$

$$
\hat{v} = \theta f + (1-\theta)v.
\tag{A4}
$$

Therefore,

$$
\hat{v} = f + (1-\theta)(m^* + u).
\tag{A5}
$$

Recall that f, m^*, and u are mutually independent. Hence,

$$
\text{cov}\left(\hat{v}, v^*\right) = \sigma^2(f) + (1-\theta)\sigma^2(m^*)
\tag{A6}
$$

and

$$
\sigma^2(\hat{v}) = \sigma^2(f) + (1-\theta)^2\left[\sigma^2(m^*) + \sigma^2(u)\right].
\tag{A7}
$$

Replacing the terms that are on the left-hand sides of Equations A6 and A7 with the expressions on the right-hand sides of these equations in the definition of $Q(\cdot)$ given in Equation A1 yields

$$
\begin{aligned}
Q(\theta) &= \ln\left[\sigma^2(f) + (1-\theta)\sigma^2(m^*)\right] \\
&\quad - \frac{1}{2}\ln\left\{\sigma^2(f) + (1-\theta)^2\left[\sigma^2(m^*) + \sigma^2(u)\right]\right\} \\
&\quad - \ln\left[\sigma^2(m^*)\right].
\end{aligned}
\tag{A8}
$$

Differentiating the right-hand side of Equation A8 produces

$$
\begin{aligned}
Q'(\theta) &= \frac{-\sigma^2(m^*)}{\sigma^2(f) + (1-\theta)\sigma^2(m^*)} \\
&\quad + \frac{(1-\theta)\left[\sigma^2(m^*) + \sigma^2(u)\right]}{\sigma^2(f) + (1-\theta)^2\left[\sigma^2(m^*) + \sigma^2(u)\right]}.
\end{aligned}
\tag{A9}
$$

A necessary condition for a given choice for θ to yield the maximum or minimum value of $Q(\theta)$ is that $Q'(\theta) = 0$. From Equation A9, we see that $Q'(\theta) = 0$ only once for finite choices of θ, when the following condition holds:

$$
1 - \theta = \frac{\sigma^2(m^*)}{\sigma^2(m^*) + \sigma^2(u)}.
\tag{A10}
$$

Hence,

$$
\theta = \frac{\sigma^2(u)}{\sigma^2(m^*) + \sigma^2(u)}
\tag{A11}
$$

is the unique finite choice for θ that could maximize $Q(\theta)$.

From Equation A9, we see that $Q'(0) \geq 0$ and $Q'(1) \leq 0$. From Equation A11, it is evident that our choice for θ is between 0 and 1. Hence, $Q'(\cdot)$ is decreasing around the choice for θ so that we have a maximum value rather a minimum.

Notes

1. I use the term "fundamental indexation" from the title of Arnott, Hsu, and Moore (2005) to refer to any indexing strategy in which fundamental values of the securities, such as earnings, book value, dividends, and/or other variables not related to market capitalization, are used for weighting the portfolio holdings. There are a number of such strategies in the market. Fundamental Index is a registered trademark of Research Affiliates, LLC.

2. As Perold (2007) noted, Siegel (2006) coined the term "noisy market hypothesis." Arnott, Hsu, and Moore (2005), Hsu (2006), and Treynor (2005) presented the ideas that this term denotes. The idea of using fundamental measures of company size as alternatives to market capitalization goes back at least to Berk (1995) and Grobowski and King (1996), albeit in a different context. Those researchers were questioning whether there is a "size effect" in stock returns apart from a market-cap effect.

3. Arnott, Hsu, Liu, and Markowitz (2007) demonstrated that noise can explain the size and value effects.

4. Treynor assumed that "MVI [market-valuation indifferent] investors spend the same number of dollars on the underpriced as they spend on the overpriced stocks" (p. 67). Of course, they can do this only if they know the relative true values of all stocks, a condition that proponents of fundamental indexation (including Treynor in the same paper) deny is necessary for their systems to work. See Perold (2007).

5. Strictly speaking, I compare the correlations of the natural logarithms of these variables. See the discussion that follows.

6. The *a priori* superior approach is the one that has the higher *expected* return. The realized returns of each strategy will, of course, differ from the expected returns.

7. Wiandt (2007) examined the recent performance of exchange-traded funds that track fundamentally weighted indices and found that that they have been lagging their market-cap-weighted counterparts. He attributed this performance lag to the value bias in the fundamentally weighted indices during a time when value is out of favor.

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